Galactic & planetary dynamics – problem set 2

1. Let $(\boldsymbol{\theta}^0, \mathbf{J}^0)$ be angle-action variables for the Hamiltonian $H_0(\mathbf{J}^0)$. We wish to construct approximate angle-action variables $(\boldsymbol{\theta}^1, \mathbf{J}^1)$ for the perturbed Hamiltonian $H = H_0 + \epsilon H_1$, where $\epsilon H_1(\boldsymbol{\theta}^0, \mathbf{J}^0)$ is a known function of the $(\boldsymbol{\theta}^0, \mathbf{J}^0)$. Explain why it is appropriate to construct these new $(\boldsymbol{\theta}^1, \mathbf{J}^1)$ by using a type-2 generating function of the form $F_2(\boldsymbol{\theta}^1, \mathbf{J}^0) = \boldsymbol{\theta}^1 \cdot \mathbf{J}^0 + \epsilon S(\boldsymbol{\theta}^1, \mathbf{J}^0)$. What is the relationship between $(\boldsymbol{\theta}^0, \mathbf{J}^0)$ and $(\boldsymbol{\theta}^1, \mathbf{J}^1)$ in terms of the function S? Obtain an expression for $H = H_0 + \epsilon H_1$ as a function of $\boldsymbol{\theta}^1, \mathbf{J}^1$, correct to first order in ϵ. By Fourier expanding

$$S(\boldsymbol{\theta}^1, \mathbf{J}^0) = \sum_{\mathbf{k}} S_{\mathbf{k}}(\mathbf{J}^0) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}^1} \tag{0.1}$$

and making a similar expansion for the perturbation H_1, explain how it is possible to choose the $S_{\mathbf{k}}(\mathbf{J}^0)$ to eliminate the $O(\epsilon)$ dependence of the Hamiltonian on the angles $\boldsymbol{\theta}^1$.

2. Consider an orbit having semimajor axis a and eccentricity e in a Kepler potential. How are the eccentric anomaly η and mean anomaly w of a point along such an orbit defined? By expressing the cartesian coordinates (x, y) of the orbit in terms of the eccentric anomaly and then integrating over the mean anomaly, or otherwise, show that

$$\langle r^2 \rangle = a^2\left(1 + \tfrac{3}{2}e^2\right), \quad \langle r^2 \cos^2\varphi \rangle = \tfrac{1}{2}a^2(1 + 4e^2), \quad \langle r^2 \sin^2\varphi \rangle = \tfrac{1}{2}a^2(1 - e^2), \tag{0.2}$$

where $\langle Q \rangle$ denotes the time average of the quantity Q.

Now consider the special case of orbits that are close to circular. Show that the mean anomaly satisfies

$$\cos w = \cos\eta + e\sin^2\eta + O(e^2) \tag{0.3}$$

and hence that the true anomaly is given by

$$\begin{aligned} \varphi &= \eta + e\sin\eta + O(e^2) \\ &= w + 2e\sin w + O(e^2). \end{aligned} \tag{0.4}$$

3. For a Kepler orbit of semimajor axis a, eccentricity e and inclination i show that

$$\left\langle \frac{a^3}{r^3} \right\rangle = \frac{1}{(1-e^2)^{3/2}}, \qquad \left\langle \frac{z^2}{r^5} \right\rangle = \frac{\sin^2 i}{2a^3(1-e^2)^{3/2}}. \tag{0.5}$$

4. A satellite orbits an axisymmetric planet of mass M, whose potential may be expanded as $\Phi(r,\theta) = -GM/r + \Phi_1$, where

$$\Phi_1(r,\theta) = \frac{GM}{r}\sum_{l=2}^{\infty} \frac{J_l R^l}{r^l} P_l(\cos\theta). \tag{0.6}$$

Here R is the mean radius of the planet, the $P_l(\cos\theta)$ are Legendre polynomials and the J_l are multipole moments. Treating Φ_1 as a perturbation and considering only its $l = 2$ term, show that the orbit-averaged perturbation Hamiltonian is given by

$$\langle H_1 \rangle = \frac{GMJ_2R^2}{4a^3(1-e^2)^{3/2}}(3\sin^2 i - 2), \tag{0.7}$$

where a, e and i are the semimajor axis, eccentricity and inclination of the satellite's orbit. By expressing this in terms of Delaunay variables, show that a, e and i are constants of motion, but that the argument of periapse ω undergoes prograde precession if $\cos i > 1/\sqrt{5}$, retrograde if $\cos i < 1/\sqrt{5}$.

5. [optional] Write a piece of computer code to map between Cartesian phase-space coordinates $(\mathbf{x}, \mathbf{v})$ and Delaunay elements. Assume that $GM = 1$. Construct an orbit integrator for the potential of the previous problem and use it to check how well the averaging approximation holds for the earth (for which $J_2 \simeq 10^{-3}$).

6. [BT3.23] The earth is flattened at the poles by its spin. Consequently orbits in its potential do not conserve total angular momentum. Many satellites are launched in inclined, nearly circular orbits only a few hundred kilometers about the Earth's surface, and their orbits must remain nearly circular, or they will enter the atmosphere and be destroyed. Why do the orbits remain nearly circular? [See Tremaine & Yavetz (2014)]

7. Consider the restricted three-body problem in which a test body orbiting a central star of unit mass is perturbed by a planet of mass m that moves on a circular orbit of unit radius.

(a) The motion of the test particle can be described by a Hamiltonian of the form $H = H_0(\mathbf{J}) + H_1(\boldsymbol{\theta}, \mathbf{J})$, in which

$$H_0(\mathbf{J}) = -1/2J_\lambda^2 + (J_\varpi - J_\lambda). \tag{0.8}$$

Explain how actions $\mathbf{J} = (J_\lambda, J_\varpi)$ and angles $\boldsymbol{\theta} = (\lambda, -\varpi)$ are related to the orbital elements (a, e, ω, w) of the test particle. What is the origin of the $(J_\varpi - J_\lambda)$ term in H_0?

(b) Write down an expression for H_1 in terms of the radius r and true anomaly φ of the test particle. Without detailed calculation explain how to expand this H_1 using Laplace coefficients $b_s^{(j)}(a)$ and and thence to angle–action coordinates $(\boldsymbol{\theta}, \mathbf{J})$.

[The Laplace coefficients $b_s^{(j)}(a)$ are defined through $(1 + a^2 - 2a\cos\phi)^{-s} = \sum_{j=0}^{\infty} b_s^{(j)}(a)\cos j\phi$.]