Galactic & planetary dynamics – problem set 1

1. What is meant by the *dynamical* time and the *relaxation* time, $t_{\rm dyn}$ and $t_{\rm relax}$ respectively, of an N-body stellar system. Identifying any assumptions that you make, show that in three-dimensional systems $t_{\rm relax}/t_{\rm dyn} \sim (0.1N/\log N)t_{\rm dyn}$. Does this relation hold for two-dimensional systems, such as stellar discs?

2. Let $(\mathbf{q},\mathbf{p})$ be canonical coordinates for a system having n degrees of freedom. Consider the transformation $\mathbf{Q}=A\mathbf{q}$, $\mathbf{P}=B\mathbf{p}$ where A and B are $n\times n$ matrices having constant coefficients. How must A and B be related to ensure that $(\mathbf{Q},\mathbf{P})$ are canonical?

Referred to an inertial coordinates $\mathbf{X}$ and momenta $\mathbf{P}$ the Hamiltonian for a system of N planets orbiting a central star is

$$H(\{\mathbf{X}\},\{\mathbf{P}\}) = \frac{\mathbf{P}_0}{2m_0} + \sum_{n=1}^{N}\frac{\mathbf{P}_n^2}{2m_n} + V, \tag{0.1}$$

where $V(\mathbf{X}_0,\mathbf{X}_1,...,\mathbf{X}_N)$ accounts for the gravitational interactions among the bodies.

(a) Consider a canonical transformation to new, heliocentric, coordinates $(\mathbf{x}_0,...,\mathbf{x}_N)$ in which $\mathbf{x}_0\equiv\mathbf{X}_0$ is the location of the sun and $\mathbf{x}_n\equiv\mathbf{X}_n-\mathbf{X}_0$ are the heliocentric coordinates of the planets $(n=1,...,N)$. What the the corresponding conjugate momenta $\mathbf{p}_0$ and $\mathbf{p}_1,...,\mathbf{p}_N$? Show that the Hamiltonian in these new coordinates is given by

$$H(\{\mathbf{x}\},\{\mathbf{p}\}) = \frac{1}{2m_0}\left(\mathbf{p}_0 - \sum_{n=1}^{N}\mathbf{p}_n\right)^2 + \sum_{n=1}^{N}\frac{\mathbf{p}_n^2}{2m_n} + V. \tag{0.2}$$

(b) consider an alternative transformation in which $\mathbf{x}_1,...,\mathbf{x}_n$ remain as in part (a), but $\mathbf{x}_0$ is instead defined to be the location of the centre of mass of the whole system. What are the momenta $\mathbf{p}_0$ and $\mathbf{p}_n$ in this case? Show that the Hamiltonian is given by

$$H(\{\mathbf{x}\},\{\mathbf{p}\}) = \frac{\mathbf{p}_0^2}{2m_{\rm tot}} + \frac{1}{2m_0}\left[\sum_{n=1}^{n}\mathbf{p}_n\right]^2 + \sum_{n=1}^{N}\frac{\mathbf{p}_n^2}{2m_n} + V, \tag{0.3}$$

where $m_{\rm tot}=m_0+\sum_{n=1}^N m_n$. What advantage, if any, does this coordinate system have over that used in part (a)?

3. [BT3.26,3.27] Consider the Hamiltonian $H=A+B$ and let $L_A\bullet\equiv[\bullet,A]$, $L_B\equiv[\bullet,B]$ denote derivatives along the phase flows produced by the functions A and B respectively. Show that the leapfrog integrator

$$\exp\left(\tfrac{1}{2}\tau L_A\right)\exp\left(\tau L_B\right)\exp\left(\tfrac{1}{2}\tau L_A\right) \tag{0.4}$$

corresponds to motion in a surrogate Hamiltonian $H_{\rm surr}=H+H_{\rm err}$, in which the error Hamiltonian

$$H_{\rm err} = \tfrac{\tau^2}{12}\left[[A,B],B+\tfrac{1}{2}A\right]+O(\tau^4). \tag{0.5}$$

Consider a new integrator composed of three successive leapfrog steps, the first of length $a\tau$, followed by one of length $b\tau$ and finally another of length $a\tau$, where $2a+b=1$. Find the constants a and b that kill off all terms lower than $O(\tau^4)$ in the error Hamiltonian of this new integrator.

4. The phase-space density $f(\mathbf{x},\mathbf{v},t)$ of stars in a galaxy evolves according to the collisionless Boltzmann equation (CBE)

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial\Phi}{\partial\mathbf{x}}\cdot\frac{\partial f}{\partial\mathbf{v}} = 0, \tag{0.6}$$

where $\Phi(\mathbf{x}, t)$ is the gravitational potential. Use invariance of Poisson brackets under canonical transformations to express the CBE in spherical polar coordinates (r, θ, ϕ).

5. [BT3.7] A test particle moves in a spherically symmetric potential $\Phi(r)$. Write down an expression for the turning points $\dot{r} = 0$ of its orbit in terms of (E, L^2), where E and L are the particle's energy and angular momentum per unit mass. Show that there are at most two such turning points when Φ is generated by a non-negative mass density $\rho(r)$. [Hint: substitute $u = 1/r$.]

6. [BT3.19, Touma & Tremaine 1997] A spherically symmetric cluster of stars has potential $\Phi(r) = Cr^\alpha$, with $-1 \le \alpha \le 2$ and $C > 0$ for $\alpha > 0$, $C < 0$ for $\alpha < 0$. Show that the ratio of radial to azimuthal periods is

$$\frac{T_r}{T_\varphi} = \begin{cases} 1/\sqrt{2+\alpha}, & \text{for nearly circular orbits,} \\ \begin{cases} \frac{1}{2}, & \text{for } \alpha > 0 \\ 1/(2+\alpha), & \text{for } \alpha < 0 \end{cases} & \text{for nearly radial orbits.} \end{cases} \tag{0.7}$$

[Hint: to calculate $\Delta\varphi$ for $L_z \simeq 0$ use $\int_1^\infty \frac{\mathrm{d}x}{x\sqrt{x^b - 1}} = \frac{\pi}{b}$ for $b > 0$. For nearly circular orbits, try a substitution of the form $r = A + B \sin\alpha$.]

7. This question [stolen from BT3.34] is intended as a (mostly) straightforward demonstration of the construction of actions for *Stäckel potentials* of the form

$$\Phi(u, v) = \frac{U(u) - V(v)}{\sinh^2 u + \sin^2 v} \tag{0.8}$$

where the confocal spheroidal coordinates (u, v) are related to the Cartesian coordinates (x, y) through $x = \Delta \sinh u \sin v$, $y = \Delta \cosh u \cos v$, and $U(u)$ and $V(v)$ are arbitrary functions.

(a) Sketch contours of constant u and contours of constant v in the (x, y) plane. What limits should we place on the range of u and v to ensure that each point on the (x, y) plane is covered just once?

(b) Obtain expressions for the momenta p_u and p_v conjugate to (u, v) and show that each may be expressed as $\Delta^2(\sinh^2 u + \sin^2 v)$ times a generalised velocity, $\dot{u}$ or $\dot{v}$. Construct the Hamiltonian $H(u, v, p_u, p_v)$.

(c) Assuming a generating function $S(u, v, \mathbf{J}) = S_u(u, \mathbf{J}) + S_v(v, \mathbf{J})$, how are p_u and p_v related to S_u and S_v? Separate the Hamilton–Jacobi equation $H(u, v, p_u, p_v) = E$ to show that $I = E \sinh^2 u - p_u^2/2\Delta^2 - U(u)$ is an integral of motion and that $E \sin^2 v + I = p_v^2/2\Delta^2 - V$.

(d) Show that $[f(a, b), A] = \frac{\partial f}{\partial a}[a, A] + \frac{\partial f}{\partial b}[b, A]$ for any sufficiently smooth functions a, b, $f(a, b)$ and A.

(e) Show that

$$I = \frac{\sinh^2 u(p_v^2 - 2\Delta^2 V) - \sin^2 v(p_u^2 + 2\Delta^2 U)}{2\Delta^2(\sinh^2 u + \sin^2 v)}, \tag{0.9}$$

and hence, or otherwise, that $[H, I] = 0$.

(f) Write down an expression for each the actions J_u and J_v as an integral over a functions that involves E, I and either $U(u)$ and $V(v)$. Hence show that J_u and J_v are involution, $[J_u, J_v] = 0$.

(g) [optional] Consider the particular case given by the functions $U(u) = -W \sinh u \tan^{-1}(\Delta \sinh u/a_3)$ and $V(v) = W \sin v \tanh^{-1}(\Delta \sin v/a_3)$ with $\Delta = 0.6a_3$. By introducing and plotting effective potentials $U_{\text{eff}}(u)$ and $V_{\text{eff}}(v)$, or otherwise, show that there are two qualitatively different families of orbits in this $\Phi(u, v)$, the choice of which depends on the sign of I. [Hint: see BT§3.5.3.]

8. [Optional, but recommended] Write a program that uses leapfrog, Runge-Kutta or similar to perform numerical orbit integration in the flattened logarithmic potential $\Phi(x, y) = \frac{1}{2}\log(1 + x^2 + y^2/q^2)$ with $q = 0.7$, explaining any tests you make to check the believeability of your results. Use your program to construct $(y, \dot{y})$ surfaces of section for different energies. Identify at least four different orbit families.