Chapter 1

Introduction

These are work-in-progress notes for the MMathPhys course "Galactic and Planetary Dynamics", which you can think of as a prequel to Michaelmas Term's "Kinetic Theory" lectures. The starting point (actually, nearly the whole course!) is the one-body problem: understanding test-particle orbits in smooth background potentials, and how they respond when perturbed. From there we build up to self-consistent, steady-state equilibrium models of galaxies and then investigate their response to various perturbations. I assume you have had some exposure to Hamiltonian mechanics. Some background in kinetic theory and astrophysics would be useful, but is not essential.

Reading: Tremaine (2023) and Morbidelli (2002) for planetary systems; Binney & Tremaine (2008) for galaxies; Arnold (1978), Fasano et al. (2013) for background in classical mechanics.

These notes are from 7th February 2025. Mistakes are guaranteed. Corrections and suggestions are welcome: `john.magorrian@physics.ox.ac.uk`.

Chapter 2

The two-body problem

Let's go back to basics and consider the two-body problem: a pair of bodies of masses m_1, m_2 at positions $\mathbf{r}_1$, $\mathbf{r}_2$ interacting with potential energy $V(\mathbf{r}_1, \mathbf{r}_2) = m_1 m_2 U(|\mathbf{r}_2 - \mathbf{r}_1|)$ so that the force between them is central. We'll shortly specialize to the choice $U(r) = -G/r$ appropriate for Newtonian gravity, but for the moment let's consider general $U(r)$.

2.1 Reduction to one-body problem

The equations of motion (EOM) for the system are

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= -\frac{\partial V}{\partial \mathbf{r}_1}, \\ m_2 \ddot{\mathbf{r}}_2 &= -\frac{\partial V}{\partial \mathbf{r}_2}. \end{aligned} \tag{2.1}$$

Introducing the the centre of mass of the pair,

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \tag{2.2}$$

and their relative displacement

$$\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \tag{2.3}$$

the EOMs can be written as

$$\begin{aligned} M\ddot{\mathbf{R}} &= \mathbf{0}, \\ \ddot{\mathbf{r}} &= -M \frac{\mathrm{d}U}{\mathrm{d}r} \hat{\mathbf{r}} \end{aligned} \tag{2.4}$$

where $M \equiv m_1 + m_2$ is the total mass of the system and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$. The first of this pair of equations is conservation of linear momentum of the system. The second shows that the two-body problem can be reduced to that a single equation for the displacement $\mathbf{r}$. Another way of writing this reduced, one-dimensional EOM is

$$\ddot{\mathbf{r}} = -\frac{\partial \Phi}{\partial \mathbf{r}}, \tag{2.5}$$

where the potential $\Phi(r) = MU(r)$.

The energy of the system,

$$\begin{aligned} E &= \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 + V \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 + \mu \Phi, \end{aligned} \tag{2.6}$$

splits into a contribution from the the kinetic energy of the centre of mass motion (first term) plus the total energy of a one-body system of reduced mass $\mu \equiv m_1 m_2/(m_1+m_2)$ moving in the potential Φ. The system's angular momentum

$$\begin{aligned}\mathbf{L} &= \mathbf{r}_1 \times m_1 \dot{\mathbf{r}}_1 + \mathbf{r}_2 \times m \dot{\mathbf{r}}_2 \\ &= \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{r} \times \mu \dot{\mathbf{r}},\end{aligned} \tag{2.7}$$

is conserved, because

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R} \times M\dot{\mathbf{R}}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r} \times \mu\dot{\mathbf{r}}\right) = \mathbf{0} \tag{2.8}$$

using the usual vector identity $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ together with $\ddot{\mathbf{R}} = \mathbf{0}$ and $\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}$ from equations (2.4). A consequence is that $\mathbf{r} \cdot \mathbf{L} =$ constant: i.e., $\mathbf{r}$ is confined to a plane whose normal is set by the direction of $\mathbf{L}$.

What does motion in this plane look like? Let us introduce plane polar coordinates (r, φ), so that $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\hat{\boldsymbol{\varphi}}$. Then the vector equation (2.5) becomes the pair of scalar equations

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}, \quad 2\dot{r}\dot{\varphi} + r^2\ddot{\varphi} = 0. \tag{2.9}$$

The second of these is conservation of angular momentum again. Let $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ be the angular momentum (per unit reduced mass) of the reduced system. Then the first equation can be written as $\ddot{r} - h^2/r^3 = -\mathrm{d}\Phi/\mathrm{d}r$. Substituting $u \equiv 1/r$ and using $\frac{\mathrm{d}}{\mathrm{d}t} = hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}$ it becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = -\frac{1}{h^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u}. \tag{2.10}$$

2.2 Newtonian gravity

Now specialise to the case $\Phi = -GM/r = -GMu$. The solution to (2.10) is then $u(\varphi) = A\cos(\varphi - \varphi_0) + GM/h^2$, where A and φ_0 are constants of integration. So $r = 1/u$ is given by

$$r(\varphi) = \frac{h^2/GM}{1 + \frac{AGM}{h^2}\cos(\phi - \phi_0)} = \frac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)}, \tag{2.11}$$

in which the constants A and h are replaced by new variables a and e. This is the equation of a conic section with focus at $r = 0$ and pericentre at $\varphi = \varphi_0$. Without loss of generality we set $\varphi_0 = 0$ from now on. If the eccentricity $e < 1$ then the orbit is bound and traces a closed ellipse with semimajor axis a. Its pericentre is at $r = a(1-e)$, $\varphi_0 = 0$, apocentre at $r = a(1+e)$, $\varphi_0 = 0$. The marginally bound $e = 1$ case is a parabola; unbound $e > 1$ orbits are hyperbolae.

Our solution (2.11) gives only the geometrical shape of the path, $r(\varphi)$, without directly telling us anything about its time dependence. To work out the latter we need to use $\mathrm{d}\varphi/\mathrm{d}t = h/r^2$, where $h^2 = GMa(1-e^2)$. From this it immediately follows that the period of bound, elliptical orbits is

$$T = 2\pi\sqrt{\frac{a^3}{GM}}, \tag{2.12}$$

independent of their eccentricity e.

To go beyond this it is helpful to express

$$r(\eta) = a(1 - e\cos\eta), \tag{2.13}$$

a much simpler function of a new, fictitious angle η, known as the **eccentric anomaly**. The real, honest-to-goodness geometrical angle φ is known as the **true anomaly**. The "dynamical" angle $w = 2\pi t/T$, where $t = 0$ corresponds to pericentre, is known as the **mean anomaly**. Both η and φ increase by 2π over the course of an orbit, but neither increases linearly with time unless $e = 0$. To map among these three angles we need the following result.

Exercise: Equating (2.11) to (2.13) and rearranging gives

$$\cos\varphi = \frac{\cos\eta - e}{1 - e\cos\eta}. \tag{2.14}$$

By considering $(1-\cos\varphi)/(1+\cos\varphi)$, show that

$$\sqrt{1-e}\tan\frac{1}{2}\varphi = \sqrt{1+e}\tan\frac{1}{2}\eta,$$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}\eta} = \frac{\sqrt{1-e^2}}{1-e\cos\eta}. \tag{2.15}$$

By conservation of angular momentum $\dot{\varphi} = h/r^2 = h/a^2(1-e\cos\eta)^2$. So,

$$t = \int \frac{\mathrm{d}\eta}{\dot{\eta}} = \int \frac{\mathrm{d}\eta}{\dot{\varphi}}\frac{\mathrm{d}\varphi}{\mathrm{d}\eta} = \frac{a^2}{h}\int \mathrm{d}\eta\,(1-e\cos\eta)^2\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}. \tag{2.16}$$

Taking $\mathrm{d}\varphi/\mathrm{d}\eta$ from (2.15) and integrating, choosing $t=0$ at pericentre,

$$t = \frac{a^2}{h}\sqrt{1-e^2}(\eta - e\sin\eta) = \frac{T}{2\pi}(\eta - e\sin\eta). \tag{2.17}$$

That is, the mean and eccentric anomalies are related via

$$w = \frac{2\pi t}{T} = \eta - e\sin\eta, \tag{2.18}$$

a result known as **Kepler's equation**.

2.3 Orbit elements

The semimajor axis a, eccentricity e and mean anomaly w are three of the six **orbital elements** that are often used to specify the phase-space position of a particle orbiting a Newtonian point mass. The other three elements are the Euler angles that describe how the orbit plane is oriented in space.

Introduce Cartesian coordinates $(x', y') = r(\cos\varphi, \sin\varphi)$ within the orbital plane. From (2.13) it follows that

$$x' = a(\cos\eta - e),$$
$$y' = a\sqrt{1-e^2}\sin\eta$$
$$z' = 0. \tag{2.19}$$

Now rotate this (x', y', z') coordinate system about the Oz' axis by angle $\omega \in [0, 2\pi)$, the **argument of peripasis**. Next rotate about the new Ox' axis by the **inclination** angle $i \in [0, \pi]$. Finally rotate about the updated Oz' axis by $\Omega \in [0, 2\pi)$, the **longitude of the ascending node**: the orbit's **nodes** are the locations where it intersects the $z=0$ plane; the ascending node has $\dot{z} > 0$, the descending node $\dot{z} < 0$. Then the three-dimensional coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix}, \tag{2.20}$$

with the dependence on a, e, w entering via (x', y') given by (2.19).

All elements but the mean anomaly w are constants of motion in the unperturbed two-body problem. Perturbations will cause the elements to evolve: at any instant the phase-space location of our reduced one-body system can be specified by a set of so-called osculating elements.

2.4 The eccentricity vector

A special feature of the Kepler potential is that it is closed. Of all spherically symmetric potentials, only it and the SHO potential have this property. By direct differentiation it is easy to confirm

that an additional integral of motion for the Kepler Hamiltonian is the **Laplace–Runge–Lenz** vector,

$$\mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{GM} - \hat{\mathbf{r}}, \tag{2.21}$$

which points in the direction of pericentre and has magnitude e. For this reason it is also called the **eccentricity vector**.

Exercise: Another conserved vector is the angular momentum $\mathbf{h}$. Show that $\mathbf{h}$ and $\mathbf{e}$ are not independent and that conservation of both of them leads to a total of five independent constants of motion: $(a, e, \omega, i, \Omega)$.

Exercises

1. Two Newtonian point particles orbit their common centre of mass, which is at rest. Show that the total energy of the system is given by

$$E = -\frac{GM\mu}{2a}, \tag{2.22}$$

where a is the semimajor axis.

2. Consider an orbit having semimajor axis a and eccentricity e. By expressing the Cartesian coordinates (x, y) of the orbit in terms of the eccentric anomaly and then integrating over the mean anomaly, or otherwise, show that

$$\langle r^2 \rangle = a^2 \left(1 + \frac{3}{2}e^2\right), \quad \langle r^2 \cos^2 \varphi \rangle = \frac{1}{2}a^2(1 + 4e^2), \quad \langle r^2 \sin^2 \varphi \rangle = \frac{1}{2}a^2(1 - e^2), \tag{2.23}$$

where $\langle Q \rangle$ denotes the time average of the quantity Q.
Now consider the special case of orbits that are close to circular. Show that the mean anomaly satisfies

$$\cos w = \cos \eta + e \sin^2 \eta + O(e^2) \tag{2.24}$$

and hence that the true anomaly is given by

$$\begin{aligned} \varphi &= \eta + e \sin \eta + O(e^2) \\ &= w + 2e \sin w + O(e^2). \end{aligned} \tag{2.25}$$

3. For a Kepler orbit of semimajor axis a, eccentricity e and inclination i show that

$$\left\langle \frac{a^3}{r^3} \right\rangle = \frac{1}{(1 - e^2)^{3/2}}, \qquad \left\langle \frac{z^2}{r^5} \right\rangle = \frac{\sin^2 i}{2a^3(1 - e^2)^{3/2}}. \tag{2.26}$$

4. A test particle orbits a point mass M located at $\mathbf{R} = \mathbf{0}$. Sketch the orbit, assuming that the particle is bound with $e \neq 0$, indicating the directions of the vectors $\mathbf{h}$, $\mathbf{e}$ and $\mathbf{h} \times \mathbf{e}$. Introduce a coordinate system centred on M and let $\hat{\mathbf{z}}$ be a unit vector in the Oz direction. What pair of special points lie on the line that passes through M in the direction of $\mathbf{h} \times \hat{\mathbf{z}}$? Then by referring to equation (2.19) and the rotations (2.20), or otherwise, explain how to calculate the elements $(a, e, \omega, i, \Omega, w)$ of the test particle given its position (x, y, z) and velocity $(\dot{x}, \dot{y}, \dot{z})$ in Cartesian coordinates.

5. Let (r, ϑ, φ) be the spherical polar coordinates of the particle in the previous problem, so that it has Cartesian coordinates $(x, y, z) = r(\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$. From (2.20) show that

$$\cot i \cot \vartheta = \sin(\varphi - \Omega), \tag{2.27}$$

a result that holds for *any* orbit that is confined to a plane whose orientation is given by the angles (Ω, i). We'll use this in §5.6 later. See also Figure 3.26 of Binney & Tremaine (2008).

6. A pair of bodies approach one another on a hyperbolic orbit with relative speed v and impact parameter b. Show that the deflection angle at the end of the encounter is given by $\pi - 2\Delta\varphi$, where

$$\tan \Delta\varphi = -\frac{bv^2}{GM}. \tag{2.28}$$

7. Show that the minimum separation $r_{\min}$ during the encounter between the bodies in the previous question satisfies

$$v^2 r_{\min}^2 + 2GMr_{\min} - b^2 v^2 = 0. \tag{2.29}$$

Suppose now that the bodies have identical masses $m_1 = m_2$ and radius a. Obtain an expression for the minimum value of b needed to avoid contact between them. Hence show that each body has an effective cross section for physical collisions of

$$\sigma_\star = \pi(2a)^2 \left(1 + \Theta\right), \tag{2.30}$$

where $\Theta = v_{\rm esc}^2/v^2$ with $v_{\rm esc}^2 = 2Gm_1/a$.

Chapter 3

More orbits

3.1 The collisionless approximation

Consider a cluster composed of N stars, each of mass m, arranged in a sphere of radius R. The typical speed v of star is then given by $v^2 = GmN/R$. Follow a typical star as it passes though the cluster. From (2.28) each interaction it makes as it passes another star leads to a nudge $\delta v \sim Gm/bv$ in the 1d components of its velocity, where b is the impact parameter of the encounter. Per crossing there are approximately $2\pi b\mathrm{d}bN/\pi R^2$ such encounters having impact parameters in the range from b and $b + \mathrm{d}b$. If we assume that these encounters are uncorrelated, then the mean-square change in velocity is given by

$$\frac{\Delta v^2}{v^2} \sim 2\pi \frac{N}{\pi R^2} \int_{b_{\min}}^{b_{\max}} (\delta v)^2 b \, \mathrm{d}b \sim \frac{1}{N} \log \Lambda, \tag{3.1}$$

where the **Coulomb logarithm** $\log \Lambda \equiv \log(b_{\max}/b_{\min})$. A natural choice of $b_{\max}$ is R, the radius of the cluster. Our approximation for δv assumes that the deflection angle in each encounter is small and so we set $b_{\min} \sim Gm/v^2$. Then $\log \Lambda \sim \log N$ and we have that

$$\frac{\Delta v^2}{v^2} \sim \frac{\log N}{N}. \tag{3.2}$$

A more refined version of this calculation (Hamilton et al., 2018) considers explicitly the contribution from different length scales. Instead of a sphere, take a three-dimensional box of side L containing our N stars, each of mass m and moving with typical speed v. Now consider a subbox of side xL within the big box. The subbox contains $N_x = x^3 N \pm \sqrt{x^3 N}$ stars. The mass within the subbox has fluctuations $\delta M_x = m\sqrt{x^3 N}$ which persist for a time $\delta t_x \sim xL/v$. A test particle at some distance rL from the subbox will experience a nudge to its velocity of

$$\delta v(r, x) \sim \frac{G\delta M_x}{rL} \delta t_x = \frac{v x^{5/2}}{r^2 \sqrt{N}}. \tag{3.3}$$

Assuming no correlations, the contribution to the mean-square change in the velocity of the test particle from all subboxes of scale x at distance r in crossing the big box is then

$$\Delta v^2(r, x) = \sum_r \underbrace{\left(\frac{v x^{5/2}}{r^2 \sqrt{N}} \right)^2}_{\delta v^2(r,x)} \underbrace{4\pi \left(\frac{r}{x} \right)^2}_{\#\text{ subboxes}} \underbrace{\frac{1}{x}}_{\#\delta t_x} \sim \frac{4\pi v^2 x^2}{N r^2}. \tag{3.4}$$

Now, keeping the subbox scale x fixed, we sum over separations $r = x, 2x, 3x, ...$, using the approximation that $\sum_{r=x}^{1} \frac{1}{r^2} \sim \frac{1}{x^2}$. The result is that

$$\Delta v^2(x) = \sum_r \Delta v^2(r, x) = \frac{1}{x^2} \frac{4\pi v^2 x^2}{N} = \frac{4\pi v^2}{N}, \tag{3.5}$$

independent of the spatial fluctuation scale x. This result is based on the assumption that the fluctuations in each subbox are statistically independent and that there are enough stars in each

subbox that the usual $\sqrt{N}$ Gaussian approximation to the Poisson fluctuations is applicable. The former breaks down as $x \to 1$, the latter as $x \to N^{-1/3}$. For small x the result also ignores the temporal correlations among subboxes as particles stream from each subbox to its neighbours: it assumes that the stars are completely reshuffled every $\delta t_x \propto x$. Hamilton et al. (2018) account for this by adding together the contribution from all scales from $x = x_{\rm min} = 1/N^{1/3}$ to $x = x_{\rm max} = 1$. There are $\sim \log(x_{\rm max}/x_{\rm min})$ such scales to include. Multiplying the constant (3.5) by this factor gives

$$\frac{\Delta v^2}{v^2} = \frac{4\pi}{3}\frac{\log N}{N}, \tag{3.6}$$

in $\sim$ agreement with (3.2).

We define the **relaxation time** to be the time taken for a star to lose all memory of its initial orbit. From the preceding expression (which accounts only for uncorrelated two-body encounters) this **two-body relaxation time** is given by

$$t_{\rm relax} \sim \frac{N}{\log N} t_{\rm dyn}, \tag{3.7}$$

where $t_{\rm dyn}$ is the **dynamical** or **crossing time**. In reality galaxies can relax faster than this (see exercises to §6 for one example), but the shot noise from two-body encounters is inevitable and sets a minimum relaxation rate.

Typical galaxies have $N \sim 10^{11}$ stars and are only $\sim 10^2$ crossing times old. Therefore the two-body perturbations to a stellar system caused by individual stars are largely unimportant, except possibly very close to the centre where the crossing time becomes short. Over timescales $t \ll t_{\rm relax}$ we can think of stars as test particles moving in a smooth background potential $\Phi(\mathbf{r}, t)$ sourced by the distribution of stars and dark matter. This is often called the **collisionless** assumption: it does not just mean that stars do not undergo physical collisions, but that they do not even encounter each other directly.

In this approximation there are no correlations among stars and each distinct population is completely described by its phase-space distribution function (DF), $f(\mathbf{x}, \mathbf{v}, t)$, which we define to be the phase-space *mass* density of that population.[1]

We make the additional approximation that stars are neither created nor destroyed. Then the 6d continuity equation becomes the **collisionless Boltzmann equation** (CBE),

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{3.8}$$

for the evolution of f. If f includes all matter in the galaxy then the potential $\Phi(\mathbf{x}, t)$ is

$$\Phi(\mathbf{x}, t) = -G\int \frac{f(\mathbf{x}', \mathbf{v}', t)\,{\rm d}^3\mathbf{x}'\,{\rm d}\mathbf{v}'}{|\mathbf{x} - \mathbf{x}'|} + \Phi_{\rm ext}(\mathbf{x}, t), \tag{3.9}$$

where $\Phi_{\rm ext}(\mathbf{x}, t)$ is any externally imposed gravitational perturbation. If there are multiple distinct populations $i = 1, ..., n$, each with its own DF f_i then each will make its own contribution to Φ.

Much of this course will be concerned with constructing solutions to the coupled pair of equations (3.8) and (3.9). We start by investigating the qualitative behaviour of orbits in plausible steady-state potentials $\Phi(\mathbf{x})$.

3.2 Orbits in spherically symmetric potentials

A test particle moving in a spherically symmetric potential $\Phi(r)$ has constant angular momentum $\mathbf{L}$ per unit mass.[2] Its motion is therefore confined to an orbital plane, $\mathbf{r}\cdot\mathbf{L} =$ constant. We use

[1] That is $f(\mathbf{x}, \mathbf{v}){\rm d}^3\mathbf{x}{\rm d}^3\mathbf{v}$ is the mass enclosed within a volume element ${\rm d}^3\mathbf{x}{\rm d}^3\mathbf{v}$ at phase-space location $(\mathbf{x}, \mathbf{v})$. An alternative convention is to take f to be a *probability* density.

[2] Unless stated otherwise we set $m = 1$ for test particles, so that their energies and momenta are all per unit notional mass.

polar coordinates (r, φ) to label its position in this plane. Conservation of energy,

$$\begin{aligned} E &= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\varphi}^2 + \Phi(r) \\ &= \frac{1}{2}\dot{r}^2 + \underbrace{\frac{L^2}{2r^2} + \Phi(r)}_{\Phi_{\rm eff}(r)} \end{aligned} \tag{3.10}$$

gives a relation between r and $\dot{r}$. The constraint that $\dot{r}^2 > 0$ means that r is confined to some range $r_- < r < r_+$, where the peri- and apo-centre radii r_- and r_+ can be determined by finding the roots of $\Phi_{\rm eff}(r) - E = 0$.

Rearranging (3.10) into an expression for $1/\dot{r} = {\rm d}t/{\rm d}r$ and integrating, the **radial period** of the particle's orbit (that is, the time taken for it complete a single loop of its $r_- \to r_+ \to r_-$ motion) is given by

$$T_r = 2\int_{r_-}^{r_+} \frac{{\rm d}r}{\sqrt{2(E - \Phi(r)) - L^2/r^2}}. \tag{3.11}$$

Meanwhile φ increases at a rate $\dot{\varphi} = L/r^2$, so that over the course of a radial period φ increases by an amount

$$\Delta\varphi = 2\int_{r_-}^{r_+} \frac{L\,{\rm d}r}{r^2\sqrt{2(E - \Phi(r)) - L^2/r^2}}. \tag{3.12}$$

The angular period is therefore $T_\varphi = 2\pi T_r/|\Delta\varphi|$. If the ratio $T_\varphi/T_r = 2\pi/|\Delta\varphi|$ is rational then the particle eventually retraces its orbit exactly: its orbit is closed.

> **Exercise:** Show that $T_\varphi/T_r = 1$ for the Kepler potential $\Phi(r) = -GM/r$ and that $T_\varphi/T_r = 2$ for the simple harmonic oscillator potential $\Phi(r) = \frac{1}{2}\Omega^2 r^2$. By considering the (three-dimensional) mass distributions that give rise to these potentials, what does this result say about the range of plausible values of T_φ/T_r in plausible astrophysical systems?

A very handy result comes from approximating the effective potential as a quadratic,

$$\Phi_{\rm eff}(r) \simeq \Phi_{\rm eff}(r_{\rm g}) + \frac{1}{2}\kappa^2(r - r_{\rm g})^2 + \cdots, \tag{3.13}$$

about its minimum, the location of which is known as the **guiding centre radius**, $r_{\rm g}$. The constant

$$\kappa^2 \equiv \left.\frac{{\rm d}^2\Phi_{\rm eff}}{{\rm d}r^2}\right|_{r_g} = \left(\frac{{\rm d}^2\Phi}{{\rm d}r^2} + \frac{3L^2}{r^4}\right)_{r_g} = \left(\frac{{\rm d}^2\Phi}{{\rm d}r^2} + \frac{3}{r}\frac{{\rm d}\Phi}{{\rm d}r}\right)_{r_g} \tag{3.14}$$

is known as the **radial epicycle frequency**. Particles that are on almost-circular orbits (r remains close to $r_{\rm g}$) undergo simple harmonic motion in the radial direction with freqency κ. Meanwhile, their angular coordinate φ increases at a rate $\Omega = L/r^2$. Because we are assuming that r is close to $r_{\rm g}$, this angular frequency Ω is given by

$$\Omega^2 = \left.\frac{1}{r}\frac{{\rm d}\Phi}{{\rm d}r}\right|_{r_g}. \tag{3.15}$$

> **Exercise:** Explain how to find the radius $r_{\rm c}$ of a circular orbit (i) that has angular momentum L or, alternatively, (ii) that has energy E. Show that the **circular speed** $v_{\rm c}$ of such orbits are given by $v_{\rm c}^2 = (r{\rm d}\Phi/{\rm d}r)\,|_{r_{\rm c}}$.

In summary, orbits in spherical potentials are straightforward. Each such orbit has four conserved integrals of motion: the energy E and the three components of angular momentum $\mathbf{L}$. The orbit is confined to a plane whose normal is set by the direction of $\mathbf{L}$. Within this plane the motion is confined to an annulus whose edges are given by the solutions to $E = \Phi_{\rm eff}(r)$. We can think of every orbit as being (possibly large, anharmonic, generalized) epicyclic oscillations about an underlying closed, circular orbit of radius $r_{\rm g}$ given by $E = \Phi_{\rm eff}(r_{\rm g})$.

The Kepler potential is a very special case of a spherical potential in which the radial and tangential frequencies are identical, $\kappa = \Omega$. Another special case is the spherical harmonic oscillator potential, $\Phi(r) = \frac{1}{2}\Omega^2 r^2$, for which $\kappa = 2\Omega$ everywhere.

3.3 Orbits in steady, nonspherical potentials

The structure of phase space in flattened potentials is much more interesting. As a simple example, let us consider the flattened singular logarithmic potential

$$\Phi(x,y) = \frac{1}{2}v_0^2 \log\left(x^2 + \frac{y^2}{q^2}\right), \tag{3.16}$$

in which the constant q specifies the axis ratio of the equipotential surfaces in the (x,y) plane and v_0 sets a velocity scale: when $q=1$ all circular orbits have speed v_0. This can be used as a crude model of the potential of a nonaxisymmetric disc, or, if we replace (x,y) by (R,z) –

$$\Phi(R,z) = \frac{1}{2}v_0^2 \log\left(R^2 + \frac{z^2}{q^2}\right) \tag{3.17}$$

– of orbits in the meridional (R,z) plane of an axisymmetric galaxy that have zero angular momentum ($L_z = 0$).

> **Exercise:** What is the mass density distribution that generates the potential (3.17)? What is its total mass? By considering a multipole expansion of the three-dimensional mass distribution $\rho(\mathbf{x})$, or otherwise, comment on the plausibility of ellipsoidal equipotential surfaces, $\Phi(\mathbf{x}) = \Phi(m)$, where $m^2 \equiv x^2 + y^2/b^2 + z^2/c^2$ and b and c are constants.

The left panel of Figure 3.1 shows a **surface of section** in the potential (3.16) with $v_0 = 1$ and $q = 0.7$. This surface of section is obtained by launching orbits from $y = 0$ with various initial $(x, \dot{x})$, choosing $\dot{y}$ so that the total energy $E = 1$, then using a numerical integrator to follow the orbit for many crossings of the $y = 0$ plane. Every time the orbit crosses this plane with $\dot{y} > 0$ we record its $(x, \dot{x})$ values, leaving the **consequents** on the figure. The panel on the right shows the three orbits that produce the black, green and red points on the surface of section. These orbits are known as "loop", "fish" and "banana" orbits, respectively. In each case the points plotted on the surface of section lie along the so-called **invariant curve** of the orbit: the longer we integrate for, the more densely we sample this curve.

> **Exercise:** By pure thought, or otherwise, construct a surface of section for orbits in the spherical potential (3.16) with $q = 1$.

The only integral of motion that we can write down in this two-dimensional flattened system ($q \neq 1$) is the total energy E. But if this were the only integral of motion orbits would be restricted only to the three-dimensional subspace $E = 1$ of four-dimensional phase-space, and the two-dimensional slice presented in Figure 3.1 should be filled densely. Instead, the restriction of the loop, fish and banana orbits to one-dimensional invariant curves indicates that each of these orbits respects an additional **isolating integral**, I_2, that reduces the dimensionality of phase space by one: the constraints on E and I_2 mean that each orbit explores a $4 - 2 = 2$-dimensional subspace of the full 4-dimensional phase space. These are examples of **regular orbits**: that is, orbits that have as many isolating integrals as there are degrees of freedom. Similarly, most orbits in most realistic axisymmetric galaxy models respect a **third integral** of motion in addition to energy E and the angular momentum L_z about the symmetry axis.

Two additional points:

1. each of the loop, banana and fish orbits can be thought of as epicyclic motion about an underlying closed **parent orbit**; they correspond to distinct **orbit families**, each of which has its own distinct I_2.

2. Not all orbits are regular. In particlar, orbits close to the boundaries between orbit families are irregular. For example, in the potential (3.16) the irregular orbits are those that do not lie on an invariant curve in the surface of section, but instead fill out the space between the gaps occupied by the major orbit families. In two-dimensional systems irregular orbits are strongly confined, but in three-dimensional galaxy potentials they can occupy significant fractions of phase space.

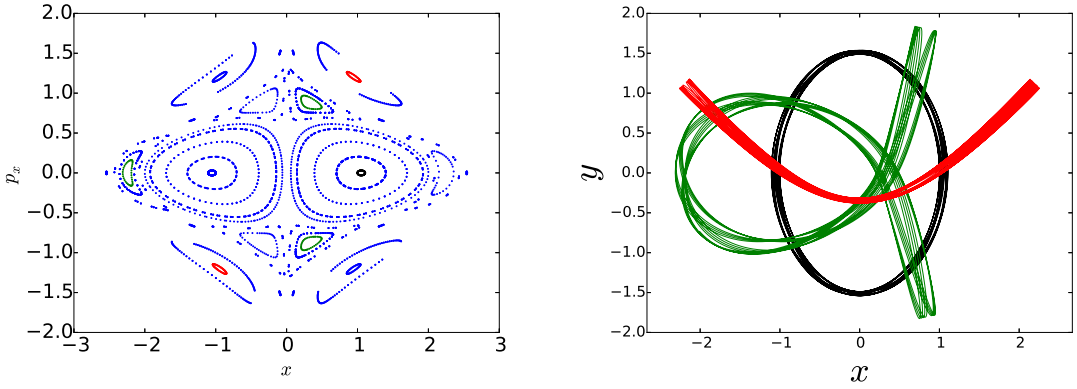

Figure 3.1: **Orbits in the flattened logarithmic potential** (3.16) with $v_0 = 1$ and $q = 0.7$. The panel on the left shows an $(x, p_x = \dot{x})$ surface of section obtained by numerically integrating orbits with energy $E = 1$. The black, red and green points are each generated by a single orbit, the (x, y) trace of which is shown on the right.

3.4 Orbits in rotating frames of reference

The next case to consider is the situation in which the potential $\Phi(\mathbf{r})$ is steady, $\partial\Phi/\partial t = 0$, in a frame that rotates with angular velocity $\boldsymbol{\Omega}_\mathrm{p}$. Recall that the EOM in such systems is (equation (4.19) later)

$$\ddot{\mathbf{r}} = -\dot{\boldsymbol{\Omega}}_\mathrm{p} \times \mathbf{r} - \underbrace{2\boldsymbol{\Omega}_\mathrm{p} \times \dot{\mathbf{r}}}_{\text{Coriolis}} - \underbrace{\boldsymbol{\Omega}_\mathrm{p} \times (\boldsymbol{\Omega}_\mathrm{p} \times \mathbf{r})}_{\text{centrifugal}} - \frac{\partial\Phi}{\partial\mathbf{r}}. \tag{3.18}$$

For slowly moving particles ($\dot{\mathbf{r}} \simeq \mathbf{0}$) in a steadily rotating potential ($\dot{\boldsymbol{\Omega}}_\mathrm{p} = \mathbf{0}$), this equation of motion is equivalent to motion in an effective potential

$$\Phi_\mathrm{eff}(\mathbf{r}) = \Phi(\mathbf{r}) - \frac{1}{2}(\boldsymbol{\Omega}_\mathrm{p} \times \mathbf{r})^2, \tag{3.19}$$

and, whether the particle moves slowly or not, it is confined to some $\Phi_\mathrm{eff} \geq E_\mathrm{J} =$ constant region (see equation (4.20) below).

Figure 3.2 shows the effective potential Φ_eff for two systems. The first is a very poor attempt at a model of a galactic bar, in which the potential

$$\Phi(x, y) = \frac{1}{2}v_0^2 \log\left(R_\mathrm{c}^2 + x^2 + \frac{y^2}{q^2}\right), \tag{3.20}$$

is stationary in a frame that rotates with constant angular velocity $\boldsymbol{\Omega}_\mathrm{p} = (0, 0, 1)$. For this plot we've chosen $R_\mathrm{c} = 0.1$ and $q = 0.8$ (see also Figure 3.14 and equation (3.103) of BT).

The second example is a three-body system consisting of a star of mass M_0 at position $\mathbf{x}_0$ and planet of mass M_1 at position $\mathbf{x}_1$ orbiting one another with constant angular velocity $\boldsymbol{\Omega}_\mathrm{p}$, where $|\boldsymbol{\Omega}_\mathrm{p}|^2 = G(M_0 + M_1)/a^3$, where $a = |\mathbf{x}_1 - \mathbf{x}_0|$ is the constant separation between the star and the planet. The effective potential is

$$\Phi_\mathrm{eff}(\mathbf{r}) = -\frac{GM_0}{\mathbf{r} - \mathbf{r}_0} - \frac{GM_1}{\mathbf{r} - \mathbf{r}_1} - \frac{1}{2}(\boldsymbol{\Omega}_\mathrm{p} \times \mathbf{r})^2 \tag{3.21}$$

We choose a corotating frame centred on the centre of mass of the system and whose z axis is parallel to $\boldsymbol{\Omega}_\mathrm{p}$ and whose x axis points towards the planet.

The extrema of Φ_eff are interesting and are called **Lagrange points**, and identified as L_1, L_2, etc, although there are different naming conventions for these points. Using the order shown in the right panel of Figure 3.2 and taking M_0 to be the sun, M_1 to be Jupiter, "Trojan" asteroids live at the trailing L_5 point, "Greeks" at the leading L_4 and "Hildas" at L_3. The Gaia and JWST

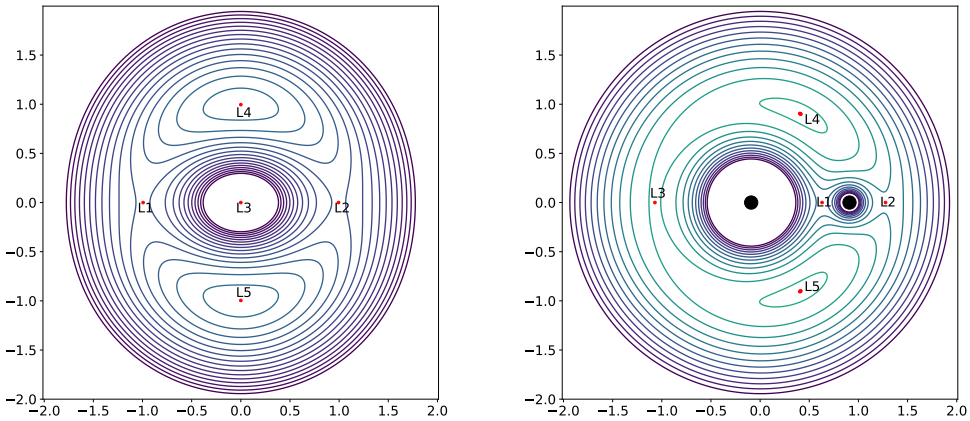

Figure 3.2: Effective potential (3.19) for the rotating flattened logarithmic potential (left) and for the restricted three-body problem (right) with $\Phi_{\rm eff}$ given by equations (3.20) and (3.21), respectively.

satellites are located at L_2 of the Sun–Earth system. They orbit the Sun with the same period as the Earth, albeit slightly further out: the additional attraction from the Earth makes this possible. The solar observatory SOHO sits at L_1. The points L_1 and L_2 are important in the theory of tides: the equipotential surface that passes through them enclosing M_1 is known as the **Roche surface** (or **Jacobi** or **Hill**).

In the rotating logarithmic potential (left panel) the gravitational and centrifugal accelerations balance exactly at the points L_1, L_2, L_4 and L_5. Stars at these points can **corotate** with the potential pattern. For this particular potential, orbits around L_3, L_4 and L_5 are stable, whereas those around L_1 and L_2 are unstable. Figure 3.3 plots some of the orbits around L_3 and L_4 in this potential.

Exercises

1. A globular cluster has mass $10^6\,M_\odot$ and half-mass radius 3 pc. Justifying any assumptions you make, calculate $t_{\rm dyn}$ and $t_{\rm relax}$. Are stellar collisions likely be important? [Hint: use equation (2.30).]

2. Does the expression (3.7) for the relaxation time apply for two-dimensional systems, such as discs? If not construct a better estimate.

3. [BT3.7] A test particle moves in a spherically symmetric potential $\Phi(r)$. Write down an expression for the turning points $\dot r = 0$ of its orbit in terms of (E, L^2), where E and L are the particle's energy and angular momentum per unit mass. Show that there are at most two such turning points when Φ is generated by a non-negative mass density $\rho(r)$. [Hint: substitute $u = 1/r$.]

4. [BT3.19, Touma & Tremaine (1997)] A spherically symmetric cluster of stars has potential $\Phi(r) = Cr^\alpha$, with $-1 \le \alpha \le 2$ and $C > 0$ for $\alpha > 0$, $C < 0$ for $\alpha < 0$. Show that the ratio of radial to azimuthal periods is

$$\frac{T_r}{T_\varphi} = \begin{cases} 1/\sqrt{2+\alpha}, & \text{for nearly circular orbits,} \\ \begin{cases} \frac{1}{2}, & \alpha > 0, \\ 1/(2+\alpha), & \alpha < 0, \end{cases} & \text{for nearly radial orbits.} \end{cases} \tag{3.22}$$

[Hint: to calculate $\Delta\varphi$ for $L_z \simeq 0$ use $\int_1^\infty \frac{{\rm d}x}{x\sqrt{x^b-1}} = \frac{\pi}{b}$ for $b > 0$. For nearly circular orbits, try a substitution of the form $r = A + B\sin\alpha$.]

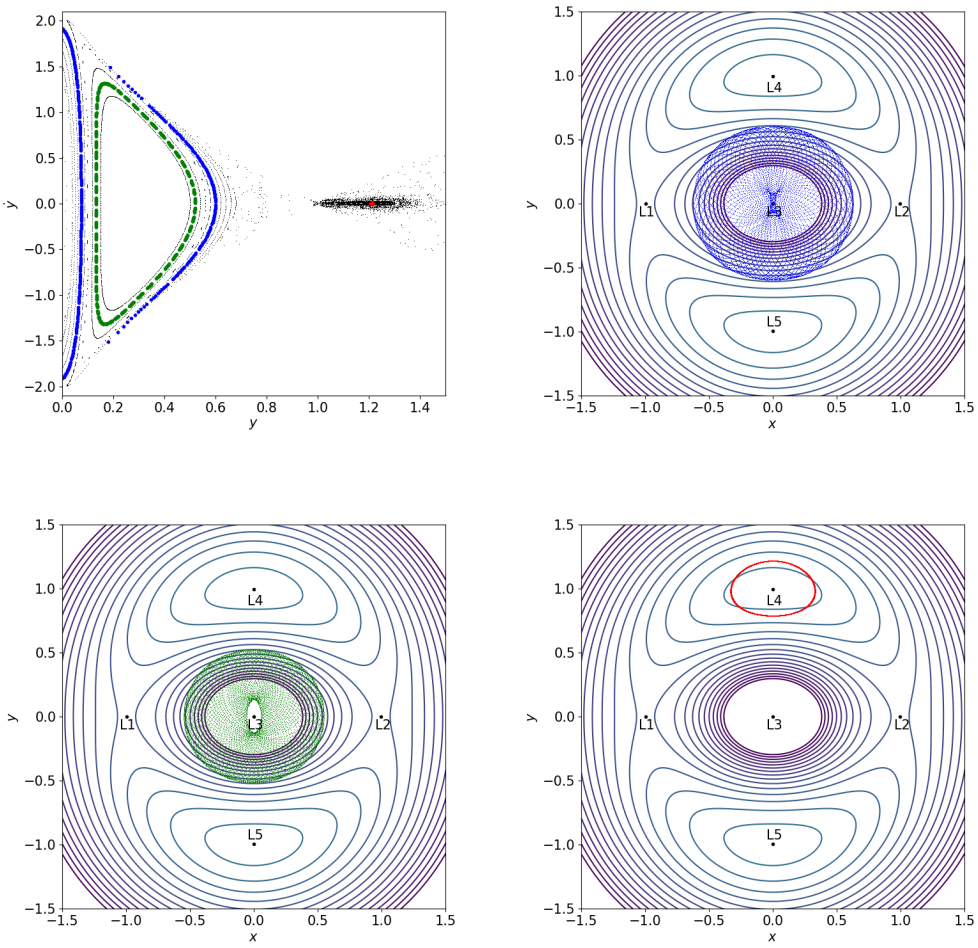

Figure 3.3: Top left: surface of section for orbits in the rotating flattened logarithmic potential (3.20) having Jacobi integral $E_{\rm J} = -0.273$ (see equation (4.20) later). The orbits corresponding to the red-, green- and blue-coloured consequents are shown in the other panels.

5. By considering contours of $\Phi_{\rm eff}$ explain why there is a volume of space within which test-particle orbits in the restricted three-body system (3.21) are bound to M_1, with the Lagrange points L_1 and L_2 lying on the boundary of this region. Show that as $M_1/M_0 \to 0$ this volume becomes a sphere of radius

$$r_{\rm H} = |\mathbf{r}_1 - \mathbf{r}_0| \left(\frac{M_1}{3M_0} \right)^{1/3} \tag{3.23}$$

(known as the **Hill radius**) centred on M_1.

6. The code that was used to generate the orbits in Figure 3.3 is given in Appendix A. By modifying this code, or otherwise, construct at least one example each of "horseshoe" and "tapole" orbits in the the restricted three-body system (3.21), giving the value of the Jacobi integral in each case. ["Tadpole" orbits execute epicyclic oscillations about either L_4 or L_5, but not both; "horseshoe" orbits occupy the horseshoe-shaped contours of $\Phi_{\rm eff}$ in the right-hand panel of Figure 3.2 that enclose all of L_3, L_4 and L_5, but not L_1 or L_2.]

Chapter 4

Recap of Hamiltonian mechanics

Orbits are most naturally described using the language of Hamiltonian mechanics, which we now review. Most of the content of this chapter will probably be familiar to you, except perhaps for §4.6 on orbit integration methods.

Recall that the arena for Lagrangian mechanics is **configuration space**, whose dimension is equal to the number of **degrees of freedom** of the system. The Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is a function of location $\mathbf{q}(t)$ within configuration space, the **generalised velocity** $\dot{\mathbf{q}}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}$ with which the location moves and possibly time.

Hamilton's principle of least action states that, if the system is at location $\mathbf{q}_0$ at time t_0 and at $\mathbf{q}_1$ at time t_1, then the path $\mathbf{q}(t)$ through configuration space between these two endpoints extremises the action integral,

$$\int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\mathrm{d}t. \tag{4.1}$$

Equivalently, the path $\mathbf{q}(t)$ satisfies the **Euler–Lagrange** (EL) equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \frac{\partial L}{\partial \mathbf{q}}, \tag{4.2}$$

subject to the boundary conditions $\mathbf{q}(t_0) = \mathbf{q}_0$, $\mathbf{q}(t_1) = \mathbf{q}_1$. Another way of writing the EL equation is by introducing the **generalised momentum**

$$\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}}, \tag{4.3}$$

so that the EL equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \frac{\partial L}{\partial \mathbf{q}}. \tag{4.4}$$

That is, the rate of change of generalised momentum is equal to the generalised force, $\frac{\partial L}{\partial \mathbf{q}}$.

How to find L? We are free to choose any L for which the EL equation (4.2) produces the correct equation of motion. For this course can always take L to be the difference between the kinetic and potential energies of the whole system. That is,

$$L = T - V, \tag{4.5}$$

in which the kinetic energy

$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}} \cdot A \cdot \dot{\mathbf{q}} + \mathbf{B} \cdot \dot{\mathbf{q}} \tag{4.6}$$

is a quadratic form in the generalised velocities $\dot{\mathbf{q}}$, and where the matrix $A(\mathbf{q}, t)$, vector $\mathbf{B}(\mathbf{q}, t)$ and the potential energy $V(\mathbf{q}, t)$, may depend on the generalised coordinates and time t, but not on the generalised velocity $\dot{\mathbf{q}}$. If our $\mathbf{q}$ are orthogonal coordinates referred to an inertial frame of reference then A is diagonal and $\mathbf{B} = 0$.

Notice from (4.1) that this L is not unique. In particular, adding any total derivative $\mathrm{d}\Lambda(\mathbf{q}, t)/\mathrm{d}t$ to L has no effect on the path $\mathbf{q}(t)$ that extremises the action integral, although it does change the definition of $\mathbf{p}$.

4.1 From Lagrangians to Hamiltonians

The EL equation (4.2) is a set of n implicit coupled second-order ODEs (4.2) for the coordinates $q_i(t)$. The equivalent form (4.4) coupled to the definition (4.3) is more appealing, but it mixes up $\mathbf{p}$'s, $\mathbf{q}$'s and $\dot{\mathbf{q}}$'s. Let's banish the latter by using a Legendre transformation to rid $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ of all generalised velocities $\dot{\mathbf{q}}$. The result of doing this is the **Hamiltonian** function

$$H(\mathbf{q}, \mathbf{p}, t) \equiv \mathbf{p} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t), \tag{4.7}$$

in which all occurences of $\dot{\mathbf{q}}$ are eliminated in favour of $\mathbf{q}$ and $\mathbf{p}$. To find the extremal path $\mathbf{q}(t)$ in terms of this new function, let's look at how this H varies with changes $(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}, \mathrm{d}t)$. Differentiating both sides of this relation we obtain

$$\begin{aligned} \frac{\partial H}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} + \frac{\partial H}{\partial \mathbf{p}} \cdot \mathrm{d}\mathbf{p} + \frac{\partial H}{\partial t}\mathrm{d}t &= \dot{\mathbf{q}} \cdot \mathrm{d}\mathbf{p} + \mathbf{p} \cdot \mathrm{d}\dot{\mathbf{q}} - \left(\frac{\partial L}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathrm{d}\dot{\mathbf{q}} + \frac{\partial L}{\partial t}\mathrm{d}t \right) \\ &= \dot{\mathbf{q}} \cdot \mathrm{d}\mathbf{p} - \frac{\partial L}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} - \frac{\partial L}{\partial t}\mathrm{d}t, \end{aligned} \tag{4.8}$$

using the Euler–Lagrange equation $\dot{\mathbf{p}} = \partial L/\partial \mathbf{q}$. This equality must hold for any $(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}, \mathrm{d}t)$. Therefore $\partial H/\partial t = -\partial L/\partial t$ and

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{4.9}$$

which are **Hamilton's equations** of motion. They are generally no easier to solve than the Euler–Lagrange equation for the system. But their first-order, explicit nature makes it easier to use them to reason about general properties of the motion.

Examples:

1. **Particle in gravitational field** Consider a particle of mass m moving in gravitational potential $\Phi(\mathbf{x}, t)$, so that its potential energy is $V(\mathbf{x}, t) = m\Phi(\mathbf{x}, t)$. The equation of motion is $\frac{\mathrm{d}}{\mathrm{d}t} m\dot{x} = -\partial V/\partial \mathbf{x}$. Taking

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2} m\dot{\mathbf{x}}^2 - V(\mathbf{x}, t), \tag{4.10}$$

the generalised momentum $\mathbf{p} \equiv \partial L/\partial \dot{\mathbf{x}} = m\dot{\mathbf{x}}$ and the Euler–Lagrange equation is $\dot{\mathbf{p}} = -\partial V/\partial \mathbf{x}$, in agreement with the usual equation. Applying the Legendre transform (4.7) to this $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ results in

$$H(\mathbf{x}, \mathbf{p}, t) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}, t), \tag{4.11}$$

for which Hamilton's equations are $\dot{\mathbf{p}} = -\partial V/\partial \mathbf{x}$ and $\dot{\mathbf{q}} = \mathbf{p}/m$.

For situations in which we have a single, test particle we may safely set $m = 1$.

2. **Point transformation: spherical polar coordinates** Expressed in spherical polar coordinates (r, ϑ, φ) the Lagrangian in the preceding example becomes (setting $m = 1$)

$$L = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2 \sin^2\vartheta\dot{\varphi}^2\right] - \Phi(\mathbf{x}, t). \tag{4.12}$$

The generalised momentum $\mathbf{p}$ has components

$$p_r = \dot{r}, \quad p_\vartheta = r^2\dot{\vartheta}, \quad p_\varphi = r^2 \sin^2\vartheta\dot{\vartheta}. \tag{4.13}$$

Applying the Legendre transformation (4.7) to this Lagrangian (4.12) yields

$$H(r, \vartheta, \varphi, p_r, p_\vartheta, p_\varphi) = \frac{1}{2}\left[p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\vartheta}\right] + \Phi(r, \vartheta, \varphi). \tag{4.14}$$

Notice that if φ does not appear explicitly in Φ then it does not appear in H either. Therefore $\partial H/\partial\varphi = 0$ and the corresponding momentum p_φ is a constant of motion. Such coordinates are known as **cyclic** coordinates. In this case the Hamiltonian is reduced to the simpler

$$H_{\text{eff}}(r, \vartheta, p_r, p_\vartheta | p_\varphi) = \frac{1}{2}\left[p_r^2 + \frac{p_\vartheta^2}{r^2}\right] + \Phi_{\text{eff}}(r, \vartheta | p_\varphi), \tag{4.15}$$

in which the **effective potential**

$$\Phi_{\text{eff}}(r, \vartheta | p_\varphi) = \Phi(r, \vartheta) + \frac{p_\varphi^2}{2r^2 \sin^2\vartheta}. \tag{4.16}$$

3. **Transform to rotating frame** When transformed to a coordinate system $\mathbf{r}$ that rotates with angular velocity $\boldsymbol{\Omega}$ with respect to the original $\mathbf{x}$ coordinate system, so that $\dot{\mathbf{x}} = \dot{\mathbf{r}} + \boldsymbol{\Omega}\times\mathbf{r}$, the Lagrangian (4.10) becomes

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}\left(\dot{\mathbf{r}} + \boldsymbol{\Omega}\times\mathbf{r}\right)^2 - \Phi(\mathbf{r}, t). \tag{4.17}$$

The generalised momentum is given by

$$\mathbf{p} = \dot{\mathbf{r}} + \boldsymbol{\Omega}\times\mathbf{r}, \tag{4.18}$$

which is the momentum $m\dot{\mathbf{x}}$ in the original, $\mathbf{x}$, frame! The Euler–Lagrange equation can be rearranged to read

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}} = -\dot{\boldsymbol{\Omega}}\times\mathbf{r} - 2\boldsymbol{\Omega}\times\dot{\mathbf{r}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) - \frac{\partial\Phi}{\partial\mathbf{r}}, \tag{4.19}$$

in which the second and third terms are the Coriolis and centrifugal forces, respectively. The Hamiltonian corresponding to the Lagrangian (4.17) is

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2} - \boldsymbol{\Omega}_{\text{p}}\cdot(\mathbf{r}\times\mathbf{p}) + \Phi(\mathbf{r}). \tag{4.20}$$

Unlike the energy E, this H has no explicit dependence on time and therefore is a constant of motion, known as the **Jacobi integral**, $E_{\text{J}} = E - \boldsymbol{\Omega}_{\text{p}}\cdot\mathbf{L} = \frac{1}{2}\dot{\mathbf{r}}^2 + \Phi_{\text{eff}}$, where $\Phi_{\text{eff}} = \Phi - \frac{1}{2}(\boldsymbol{\Omega}_{\text{p}}\times\mathbf{r})^2$ (equation 3.19).

4. **N-body system** Consider a system of N particles having masses $m_1, ..., m_N$ located at $\mathbf{x}_1, ..., \mathbf{x}_N$, in which the pairwise interparticle interaction is described by potential energy functions $V_{nl}(|\mathbf{x}_n - \mathbf{x}_l|)$ with $V_{ln} = V_{nl}$. The equations of motion are then

$$\frac{\mathrm{d}}{\mathrm{d}t}m_n\dot{\mathbf{x}}_n = -\frac{\partial}{\partial\mathbf{x}_n}\sum_{l=1}^{N} V_{nl}(|\mathbf{x}_n - \mathbf{x}_l|). \tag{4.21}$$

These equations of motion can be reproduced by the Lagrangian

$$L(\{\mathbf{x}_1, ..., \mathbf{x}_N\}, \{\dot{\mathbf{x}}_1, ..., \dot{\mathbf{x}}_N\}) = \frac{1}{2}\sum_{n=1}^{N} m_n\dot{\mathbf{x}}_n^2 - \frac{1}{2}\sum_{n=1}^{N}\sum_{l=1}^{N} V_{nl}(|\mathbf{x}_n - \mathbf{x}_l|), \tag{4.22}$$

for which the generalised momentum is the vector $\{\mathbf{p}_1, ..., \mathbf{p}_N\} = \{m_1\dot{\mathbf{x}}_1, ..., m_N\dot{\mathbf{x}}_N\}$. The Hamiltonian is

$$H(\{\mathbf{x}_1, ..., \mathbf{x}_N\}, \{\mathbf{p}_1, ..., \mathbf{p}_N\}) = \sum_{n=1}^{N}\frac{\mathbf{p}_n^2}{2m_n} + \frac{1}{2}\sum_{n=1}^{N}\sum_{l=1}^{N} V_{nl}(|\mathbf{x}_n - \mathbf{x}_l|). \tag{4.23}$$

5. **Solar system** Now consider a system of N planets about the sun. Let $\mathbf{r}_0 = \mathbf{x}_0$ be the coordinates of the sun and let $\mathbf{r}_n \equiv \mathbf{x}_n - \mathbf{x}_0$ be the coordinates of the n^{th} planet relative to the sun. Then the Lagrangian of the $(N+1)$-body system in these coordinates is

$$L(\{\mathbf{r}_0, ..., \mathbf{r}_N\}, \{\dot{\mathbf{r}}_0, ..., \dot{\mathbf{r}}_N\}) = \frac{1}{2}m_0\dot{\mathbf{r}}_0^2 + \frac{1}{2}\sum_{n=1}^{N} m_n(\dot{\mathbf{r}}_n + \dot{\mathbf{r}}_0)^2 - V \tag{4.24}$$

where we have written

$$V \equiv \frac{1}{2}\sum_{n=0}^{N}\sum_{l=1}^{N} V_{nl}(|\mathbf{r}_n - \mathbf{r}_l|) + \sum_{n=1}^{N} V_{n0}(\mathbf{r}_n) \tag{4.25}$$

for the total potential energy of the system. From (4.24) we obtain that the momenta conjugate to the coordinates $\mathbf{r}_i$ are

$$\begin{aligned} \mathbf{p}_0 &= m_0\dot{\mathbf{r}}_0 + \sum_{n=1}^{N} m_n(\dot{\mathbf{r}}_0 + \dot{\mathbf{r}}_n), \\ \mathbf{p}_n &= m_n(\dot{\mathbf{r}}_0 + \dot{\mathbf{r}}_n), \qquad n \neq 0. \end{aligned} \tag{4.26}$$

Notice that for the planets ($n \neq 0$) $\mathbf{p}_n$ is just $m_n\dot{\mathbf{x}}_n$, which is the momentum referred to the original frame. Rearranging these, we find that $m_0\dot{\mathbf{r}}_0 = \mathbf{p}_0 - \sum_{n=1}^{N}\mathbf{p}_n$ and $m_n\dot{\mathbf{r}}_n = \mathbf{p}_n - \frac{m_n}{m_0}\left(\mathbf{p}_n - \sum_l \mathbf{p}_l\right)$. Therefore

$$\begin{aligned} H(\{\mathbf{r}_n\},\{\mathbf{p}_n\}) &= \sum_{n=0}^{N} \mathbf{p}_n\cdot\dot{\mathbf{r}}_n - L \\ &= \frac{\mathbf{p}_0^2}{2m_0} - \frac{1}{m_0}\mathbf{p}_0\cdot\sum_{n=1}^{N}\mathbf{p}_n + \sum_{n=1}^{N}\frac{\mathbf{p}_n^2}{2m_n} + \frac{1}{2m_0}\sum_{n,l=1}^{N}\mathbf{p}_n\cdot\mathbf{p}_l + V \\ &= \frac{\mathbf{p}_0^2}{2m_0} - \frac{1}{m_0}\mathbf{p}_0\cdot\sum_{n=1}^{N}\mathbf{p}_n + \sum_{n=1}^{N}\mathbf{p}_n^2\left[\frac{1}{2m_n}+\frac{1}{2m_0}\right] + \sum_{n=1}^{N}\sum_{l=1}^{n-1}\frac{\mathbf{p}_n\cdot\mathbf{p}_l}{m_0} + V \end{aligned} \tag{4.27}$$

We may as well set $\mathbf{p}_0 = 0$ because $\mathbf{r}_0$ is a cyclic coordinate. Then H splits into H_0, a sum of N one-body Hamiltonians for particles having reduced masses $\mu_n = m_0m_n/(m_0+m_n)$ in orbit about the sun, plus H_1, which mops up the planet–planet interactions and the indirect terms that account for the noninertial frame:

$$\begin{aligned} H &= H_0 + H_1, \\ H_0 &= \sum_{n=1}^{N}\left[\frac{\mathbf{p}_n^2(m_0+m_n)}{2m_0m_n} + V_{0n}(\mathbf{r}_n)\right], \\ H_1 &= \sum_{n=1}^{N}\sum_{l=1}^{n-1}\left[\frac{\mathbf{p}_n\cdot\mathbf{p}_l}{m_0} + V_{nl}(\mathbf{r}_n - \mathbf{r}_l)\right]. \end{aligned} \tag{4.28}$$

Going back to the middle line of the expression (4.27) for H, a more symmetric way of writing this is

$$H = \sum_{n=1}^{N}\left[\frac{\mathbf{p}_n^2}{2m_n} + V_{0n}(\mathbf{r}_n)\right] + \frac{1}{2m_0}\left[\sum_{n=1}^{N}\mathbf{p}_n\right]^2 + \sum_{n=1}^{N}\sum_{l=1}^{n-1} V_{nl}(\mathbf{r}_n - \mathbf{r}_l), \tag{4.29}$$

which splits H into a sum of three terms, each of which is easy to integrate on its own.

Notice that in many of the cases above we have $H = T+V$, the sum of the kinetic and potential energies of the system.

Exercise: Show that if $L = T - V$ in which $T = \frac{1}{2}\sum A_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j$ is a homogenenous quadratic form in the velocities $\dot{q}_i$ and $V = V(\mathbf{q},t)$ is independent of the velocities, then $H = T + V$, with all $\dot{q}_i$ replaced by functions of $\mathbf{p}$. What happens if we add a term $\sum_i B_i(\mathbf{q},t)\dot{q}_i$ to T?

4.2 Phase space and extended phase space

A Hamiltonian that has no explicit time dependence – $\partial H/\partial t = 0$ – is called **autonomous**. For autonomous Hamiltonians we can think of the system as a point having coordinates $(\mathbf{q}(t),\mathbf{p}(t))$

moving in $2n$-dimensional **phase space**, where n is the number of degrees of freedom of the system. The "velocity" $(\dot{\mathbf{q}}(t), \dot{\mathbf{p}}(t))$ of this point is given by Hamilton's equations (4.9). As Hamilton's equations are first order, the evolution of the system is completely determined by its initial phase-space location.

Nonautonomous Hamiltonians can be made autonomous by introducing extra variables. For example, suppose that $H = H(\mathbf{q}, \mathbf{p}, t)$. Consider the new Hamiltonian

$$H'(\mathbf{q}, \mathbf{p}, \tau, E) \equiv H(\mathbf{q}, \mathbf{p}, \tau) + (-E), \tag{4.30}$$

on an extended $(2n + 2)$-dimensional phase space that includes an additional coordinate τ with corresponding generalised momentum $-E$. Here τ is the "physical" time coordinate, whereas t is used to parametrize trajectories in phase space: all occurences of t in the original $H(\mathbf{q}, \mathbf{p}, t)$ are replaced by τ. Hamilton's equations for τ and $-E$ are $\dot{\tau} = \partial H/\partial(-E) = 1$ and $-\dot{E} = -\partial H/\partial\tau$. So, τ is directly proportional to t and the autonomous Hamiltonian $H'(\mathbf{q}, \mathbf{p}, \tau, E)$ produces the same trajectories in $(\mathbf{q}, \mathbf{p})$ space as $H(\mathbf{q}, \mathbf{p}, t)$.

4.3 Poisson brackets

The **Poisson bracket** $[A, B]$ of the pair of functions $A(\mathbf{q}, \mathbf{p})$, $B(\mathbf{q}, \mathbf{p})$ is defined as

$$[A, B] \equiv \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}}. \tag{4.31}$$

Remembering that the q_i and p_i are independent coordinates labelling points in phase space, it immediately follows that

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0 \quad \text{and} \quad [q_i, p_j] = \delta_{ij}, \tag{4.32}$$

which are known as the **canonical commutation** relations or **fundamental Poisson brackets**. Similarly, from the definition (4.31) the Poisson bracket has the following properties:

$$\begin{aligned}
\text{antisymmetry:} \quad & [A, B] = -[B, A]; \\
\text{linearity:} \quad & [\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]; \quad (\alpha,\ \beta \text{ constants}) \\
\text{chain rule:} \quad & [AB, C] = [A, C]B + A[B, C]; \\
\text{Jacobi identity:} \quad & [[A, B], C] + [[B, C], A] + [[C, A], B] = 0.
\end{aligned} \tag{4.33}$$

Hamilton's equations (4.9) can be written in Poisson bracket form as

$$\dot{q}_i = [q_i, H], \quad \dot{p}_i = [p_i, H]. \tag{4.34}$$

Another way of writting the Poisson bracket (4.31) is as

$$[A, B] = \left(\frac{\partial A}{\partial \mathbf{w}}\right)^{\mathrm{T}} J \left(\frac{\partial B}{\partial \mathbf{w}}\right), \tag{4.35}$$

in which the phase-space coordinates are gathered into the $2n$-dimensional column vector

$$\mathbf{w} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \tag{4.36}$$

and the $2n \times 2n$ **symplectic unit matrix** is given by

$$J = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \tag{4.37}$$

with $\mathbf{0}_n$ and $\mathbf{1}_n$ being the $n \times n$ zero and identity matrices, respectively.

Notice that $J^T = -J$, $J^2 = -\mathbf{1}_{2n}$ and that $\det J = 1$. Any matrix A that satisfies $J = AJA^T$ is symplectic.

The fundamental Poisson brackets (4.32) become simply

$$[w_i, w_j] = J_{ij} \tag{4.38}$$

and Hamilton's equations become

$$\dot{w}_i = [w_i, H] = J_{i\alpha} \left(\frac{\partial H}{\partial w_\alpha}\right). \tag{4.39}$$

4.4 Phase flow

Any well-behaved function $B(\mathbf{q},\mathbf{p})$ defines a flow $(\mathbf{q}(\lambda),\mathbf{p}(\lambda))$ in phase space, with "velocity" vectors given by Hamilton's equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\lambda}=\frac{\partial B}{\partial \mathbf{p}},\quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\lambda}=-\frac{\partial B}{\partial \mathbf{p}},\quad \text{or}\quad \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\lambda}=J\frac{\partial B}{\partial \mathbf{w}}, \tag{4.40}$$

in which we take B as the Hamiltonian and use λ to parameterise location along the flow. The trajectories $\mathbf{w}(\lambda)=(\mathbf{q}(\lambda),\mathbf{p}(\lambda))$ for different choices of initial condition $\mathbf{w}_0=(\mathbf{q}_0,\mathbf{p}_0)$ are the **integral curves** of the function B. The Poisson bracket $[A,B]$ is the rate of change of the function $A(\mathbf{q},\mathbf{p})=A(\mathbf{w})$ as it is carried along the phase flow generated by B, because

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}\lambda}&=\frac{\partial A}{\partial \mathbf{w}}\cdot\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\lambda}=\frac{\partial A}{\partial z_i}J_{i\alpha}\frac{\partial B}{\partial z_\alpha}\quad\text{(using (4.40))}\\&=[A,B].\end{aligned} \tag{4.41}$$

In an autonomous system, any function $I(\mathbf{q},\mathbf{p})$ whose phase flow commutes with that of the Hamiltonian, $[I,H]=0$, is an **integral of motion**, $\frac{\mathrm{d}I}{\mathrm{d}t}=0$. In particular, the Hamiltonian itself is one such integral. Much of this course will be about finding a further $n-1$ integrals, when they exist.

Exercise: For a system having Hamiltonian $H(\mathbf{q},\mathbf{p},t)$ show that the rate of change of a function $f(\mathbf{q},\mathbf{p},t)$ is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+[f,H]. \tag{4.42}$$

Exercise: Suppose that $I_1,...,I_k$ are integrals of motion for the Hamiltonian H. Show that any function $f(I_1,...,I_k)$ that depends only these integrals is itself another integral of motion.

Having established that any function $B(\mathbf{q},\mathbf{p})$ defines a flow in phase space, let us return to the properties of such flows but using H for the function that generates the flow and t to parametrise position along the flow. We can have some formal fun with this for functions $f(\mathbf{q},\mathbf{p})$. Introduce the operator $\mathcal{L}_H$ defined by

$$\mathcal{L}_H\bullet\equiv[\bullet,H]. \tag{4.43}$$

Then

$$\mathcal{L}_Hf=[f,H]=\frac{\mathrm{d}f}{\mathrm{d}t},\quad \mathcal{L}_H^2f=[[f,H],H]=\frac{\mathrm{d}^2f}{\mathrm{d}t^2},\quad ...,\quad \mathcal{L}_H^nf=\frac{\mathrm{d}^nf}{\mathrm{d}t^n}, \tag{4.44}$$

in which $\mathcal{L}_H^n\bullet$ means apply the $[\bullet,H]$ operator n times. The formal Taylor series expansion

$$f(t)=\sum_{n=0}^{\infty}\frac{1}{n!}t^n\left.\frac{\mathrm{d}^nf}{\mathrm{d}t^n}\right|_{t=0} \tag{4.45}$$

can be written as the **Lie series**

$$\begin{aligned}f(t)&=\sum_{n=0}^{\infty}\frac{1}{n!}t^n\mathcal{L}_H^nf(0)\\&=\exp(t\mathcal{L}_H)f(0),\end{aligned} \tag{4.46}$$

which will be useful later.

The flows defined by Hamilton's equations (4.9), (4.40) or (4.46) have a number of special properties that sets them apart from other dynamical systems.

Liouville's theorem Consider motion from time t to time $t+\delta t$, during which $\mathbf{w}$ increases by $\delta\mathbf{w}$. To first order in δt, the Jacobian of this mapping from $\mathbf{w}$ to $\mathbf{w}+\delta\mathbf{w}$ is the $2n\times 2n$ matrix having elements

$$\frac{\partial}{\partial w_j}(w_i+\delta w_i)=\delta_{ij}+\frac{\partial}{\partial w_j}\left(J_{ik}\frac{\partial H}{\partial w_k}\right)\delta t=\delta_{ij}+A_{ij}\delta t, \tag{4.47}$$

where $A_{ij} = -A_{ij} \equiv J_{ik}\frac{\partial^2 H}{\partial w_j \partial w_k}$. Using the relation $\det(I + A\delta t) \simeq \mathrm{e}^{\mathrm{tr}(A\delta t)}$ together with the antisymmetry of A_{ij}, it follows that the determinant of this Jacobian is $+1$. Therefore the phase flow preserves volume and orientation.

Poincare invariants Volume is just one invariant of phase flows. There is a hierarchy of conserved quantities, the must fundamental of which are the integrals

$$I \equiv \oint_{\gamma(t)} \mathbf{p}\cdot \mathrm{d}\mathbf{q} = \sum_i \int_{S(t)} \mathrm{d}q_i \mathrm{d}p_i, \tag{4.48}$$

in which $\gamma(t)$ is any closed curve in phase space that moves with the phase flow. To show that I is conserved, let λ be a periodic variable parametrising position along the curve. Then

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \oint_{\gamma(t)} \left[\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\lambda} + \mathbf{p}\cdot\frac{\partial^2\mathbf{q}}{\partial\lambda\partial t}\right]\mathrm{d}\lambda. \tag{4.49}$$

Integrating the second factor of the second term by parts this becomes

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \oint \left[\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\lambda} - \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right]\mathrm{d}\lambda \\ &= -\oint \left[\frac{\partial H}{\partial \mathbf{q}}\cdot\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\lambda} + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\lambda}\cdot\frac{\partial H}{\partial\mathbf{p}}\right]\mathrm{d}\lambda = -\oint \frac{\mathrm{d}H}{\mathrm{d}\lambda}\mathrm{d}\lambda = 0.\end{aligned} \tag{4.50}$$

In fact, these invariants can be taken to be the the defining properties of phase flows: if (4.48) is conserved for all curves $\mathcal{C}(t)$ then the flow *must* be Hamiltonian (4.9).

4.5 Canonical maps

The coordinates $(\mathbf{q},\mathbf{p})$ we use to label points in phase space are not unique. There are, however, certain preferred sets of coordinates, somewhat analogous to orthonormal coordinates in everyday three-dimensional space. Suppose that we transform to new phase space coordinates $(\mathbf{Q},\mathbf{P})$ with $Q_i = Q_i(\mathbf{q},\mathbf{p})$ and similarly for P_i, $i = 1,...,n$. As before, let us write $\mathbf{W}$ for the $2n$-dimensional column vector having elements $Q_1,...,Q_n$ followed by $P_1,...,P_n$. Then $\mathbf{W} = \mathbf{W}(\mathbf{w})$. If the new $\mathbf{W} = (\mathbf{Q},\mathbf{P})$ satisfy the canonical commutation relations (4.32)

$$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0 \quad \text{and} \quad [Q_i, P_j] = \delta_{ij}, \tag{4.51}$$

or, equivalently,

$$[W_i, W_j] = J_{ij}, \tag{4.52}$$

then the new coordinates $(\mathbf{Q},\mathbf{P})$ are **canonical coordinates** and the mapping between the two sets of coordinates is called canonical or **symplectic**.

All Poisson brackets are preserved under canonical maps. To see this, introduce the Jacobian matrix

$$\left(\frac{\partial\mathbf{W}}{\partial\mathbf{w}}\right)_{ij} \equiv \frac{\partial W_i}{\partial w_j} \tag{4.53}$$

so that

$$\frac{\partial A}{\partial w_i} = \frac{\partial A}{\partial W_k}\frac{\partial W_k}{\partial w_i} \tag{4.54}$$

can be written as the column vector

$$\frac{\partial A}{\partial \mathbf{w}} = \left(\frac{\partial\mathbf{W}}{\partial\mathbf{w}}\right)^{\mathrm{T}}\frac{\partial A}{\partial\mathbf{W}}. \tag{4.55}$$

and the canonical commutation relation (4.52) becomes

$$\left(\frac{\partial\mathbf{W}}{\partial\mathbf{w}}\right) J \left(\frac{\partial\mathbf{W}}{\partial\mathbf{w}}\right)^{\mathrm{T}} = J. \tag{4.56}$$

Then

$$[A,B]_{\mathbf{w}} = \left(\frac{\partial A}{\partial \mathbf{w}}\right)^{\mathrm{T}} J \left(\frac{\partial B}{\partial \mathbf{w}}\right) = \left(\frac{\partial A}{\partial \mathbf{W}}\right)^{\mathrm{T}} \underbrace{\left(\frac{\partial \mathbf{W}}{\partial \mathbf{w}}\right) J \left(\frac{\partial \mathbf{W}}{\partial \mathbf{w}}\right)^{\mathrm{T}}}_{=J} \left(\frac{\partial B}{\partial \mathbf{W}}\right) = [A,B]_{\mathbf{W}} \tag{4.57}$$

So, Poisson brackets are independent of canonical basis.

Similarly, using (4.39) the rate of change of the coordinate W_i is

$$\begin{aligned} \dot{W}_i &= \frac{\partial W_i}{\partial w_\alpha}\dot{w}_\alpha = \frac{\partial W_i}{\partial w_\alpha} J_{\alpha\beta} \frac{\partial H}{\partial w_\beta} \\ &= [W_i, H]. \end{aligned} \tag{4.58}$$

So Hamilton's equations in the new canonical coordinates are simply

$$\dot{W}_i = [W_i, H]. \tag{4.59}$$

So, canonical maps preserve the form of Hamilton's equations of motion.

An important example of a canonical map is the mapping $\mathbf{w}_0 \to \mathbf{w}(t)$ obtained by simply drifting along under the flow generated by a Hamiltonian function $H(\mathbf{w})$. Consider motion for an infinitesmal time δt. The coordinate $\mathbf{w}_i$ changes by

$$\delta w_i = J_{i\alpha} \frac{\partial H}{\partial w_\alpha} \delta t, \tag{4.60}$$

so that $[w_i, w_j]$ is mapped to

$$[w_i + \delta w_i, w_j + \delta w_j] = [w_i, w_j] + [\delta w_i, w_j] + [w_i, \delta w_j] + [\delta w_i, \delta w_j]. \tag{4.61}$$

Exercise: Using the definition of the Poisson bracket in the form (4.35) or otherwise, show that all the $O(\delta t)$ terms in (4.61) vanish and therefore that the fundamental Poisson bracket $[w_i, w_j]$ is conserved: **time evolution is a canonical map**. More generally, the phase flow (4.40) is canonical.

Phase-space volume element In the new coordinates the phase-space volume element

$$\mathrm{d}^{2n}\mathbf{W} = \left|\det \frac{\partial \mathbf{W}}{\partial \mathbf{w}}\right| \mathrm{d}^{2n}\mathbf{w}. \tag{4.62}$$

But the condition (4.56) means that the modulus of the determinant is unity and therefore that

$$\mathrm{d}^{2n}\mathbf{W} = \mathrm{d}^{2n}\mathbf{w}. \tag{4.63}$$

Poincare invariants Let S be any two-dimensional surface in phase space and let γ be its boundary. By Green's theorem we can write the Poincare invariant (4.48) as the sum of the projections of this surface onto the $(\mathbf{Q}_i, \mathbf{P}_i)$ planes:

$$I = \oint_\gamma \mathbf{P}\cdot \mathrm{d}\mathbf{Q} = \sum_i \int_S \mathrm{d}Q_i \mathrm{d}P_i. \tag{4.64}$$

Now let (u, v) be any coordinates labelling position on the surface S. Then (4.64) becomes

$$\sum_i \int_S \frac{\partial(Q_i, P_i)}{\partial(u,v)} \mathrm{d}u\mathrm{d}v = \sum_i \int_{S_i} \left(\frac{\partial Q_i}{\partial u}\frac{\partial P_i}{\partial v} - \frac{\partial P_i}{\partial u}\frac{\partial Q_i}{\partial v}\right) \mathrm{d}u\mathrm{d}v. \tag{4.65}$$

Notice that

$$\begin{aligned} \sum_i \left(\frac{\partial Q_i}{\partial u}\frac{\partial P_i}{\partial v} - \frac{\partial P_i}{\partial u}\frac{\partial Q_i}{\partial v}\right) &= \sum_{\alpha\beta} \frac{\partial W_\alpha}{\partial u} J_{\alpha\beta} \frac{\partial W_\beta}{\partial v} \\ &= \sum_{\alpha\beta} \frac{\partial W_\alpha}{\partial w_\gamma}\frac{\partial w_\gamma}{\partial u} J_{\alpha\beta} \frac{\partial W_\beta}{\partial w_\delta}\frac{\partial w_\delta}{\partial v} \\ &= \sum_{\alpha\beta} \frac{\partial w_\gamma}{\partial u} J_{\gamma\delta} \frac{\partial w_\delta}{\partial v}, \end{aligned} \tag{4.66}$$

using the relation (4.56) to obtain the last line. Therefore I from (4.64) becomes

$$I = \sum_i \int_S \mathrm{d}Q_i \mathrm{d}P_i = \sum_{\alpha\beta} \int_S \frac{\partial W_\alpha}{\partial u} J_{\alpha\beta} \frac{\partial W_\beta}{\partial v} = \sum_{\alpha\beta} \int_S \frac{\partial w_\alpha}{\partial u} J_{\alpha\beta} \frac{\partial w_\beta}{\partial v} = \sum_i \int_S \mathrm{d}q_i \mathrm{d}p_i. \tag{4.67}$$

Conversely, any mapping that preserves all Poincare integral invariants automatically preserves all Poisson brackets.

4.6 Symplectic integrators

Hamilton's equations, $\dot{\mathbf{w}} = J\left(\frac{\partial H}{\partial \mathbf{w}}\right)$, are a set of $2n$ first-order coupled ODEs. Given phase-space coordinates $\mathbf{w}(0)$ at time $t = 0$, it is straightforward to take an off-the-shelf numerical ODE integrator to integrate these equations forwards in time to find the phase-space trajectory $\mathbf{w}(t)$ of the system at subsequent times t. Two important examples are the case of a single particle moving in potential $\Phi(\mathbf{x})$, so that $H = \mathbf{p}^2/2m + m\Phi(\mathbf{x})$, and the N-body problem (4.23).

A well-known example of a generic integrator is the fourth-order Runge–Kutta scheme. Given a timestep interval τ, the phase-space position at each new timestep is calculated from the previous one as

$$\mathbf{w}(t+\tau) \simeq \mathbf{w}(t) + \frac{\tau}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right), \tag{4.68}$$

in which the coefficients $\mathbf{k}_i(t, \mathbf{w}(t))$ are given by

$$\begin{aligned}
\mathbf{k}_1 &= \dot{\mathbf{w}}(t, \mathbf{w}), \\
\mathbf{k}_2 &= \dot{\mathbf{w}}\left(t + \frac{\tau}{2}, \mathbf{w} + \frac{\tau}{2}\mathbf{k}_1\right), \\
\mathbf{k}_3 &= \dot{\mathbf{w}}\left(t + \frac{\tau}{2}, \mathbf{w} + \frac{\tau}{2}\mathbf{k}_2\right), \\
\mathbf{k}_2 &= \dot{\mathbf{w}}\left(t + \tau, \mathbf{w} + \tau\mathbf{k}_3\right),
\end{aligned} \tag{4.69}$$

with the values of the function $\dot{\mathbf{w}}(t, \mathbf{w})$ on the RHS obtained by numerically evaluating the phase-space tangent vector $(\partial H/\partial \mathbf{q}, -\partial H/\partial \mathbf{p})$ at the appropriate point. The error in this estimate of $\mathbf{w}(t+\tau) - \mathbf{w}(t)$ turns out to be $O(\tau^5)$. Therefore the result is correct to $O(\tau^4)$ and the method is said to be "fourth order". Making the timestep τ smaller results in more accurate orbits $\mathbf{w}(t)$, and one can devise adaptive timestep schemes that estimate the error in $\mathbf{w}(t+\tau) - \mathbf{w}(t)$ and adjust τ accordingly. In the single-particle case such schemes allow us to use large steps where $V(\mathbf{x})$ is changing slowly, saving most of the computation time for locations where the gravitational force varies rapidly. Similarly, there are many more refined, higher-order integrators than this simple fourth-order Runge–Kutta one.

For Hamiltonian systems there is a special class of orbit integrator that integrates Hamilton's equations *exactly*, albeit for a surrogate Hamiltonian $H_{\rm surr}$ that is "close" to the desired Hamiltonian H. The advantage of these **symplectic integrators** is that, unlike the schemes above, they preserve Poincare invariants. To understand them, apply (4.46) to obtain the mapping

$$\mathbf{w}(t+\tau) = \exp[\tau L_H]\mathbf{w}(t), \tag{4.70}$$

where $L_H \bullet \equiv [\bullet, H]$ as before. If we could carry this out exactly, this would be a perfect stepping algorithm.

Now suppose that the Hamiltonian $H = A + B$ is a sum of two terms, A and B, for each of which we can integrate the equations of motion exactly. For example, the single-particle Hamiltonian $H = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x})$ splits into the free-particle Hamiltonian $A = \mathbf{p}^2/2m$ and the potential energy $B = V(\mathbf{x})$. Under A alone the particle simply *drifts* with $\dot{\mathbf{x}} = \mathbf{p}/m$, $\dot{\mathbf{p}} = 0$, whereas under B alone the particle is *kicked* with $\mathbf{p}$ increasing at the constant rate $-\partial V/\partial \mathbf{x}$ while $\mathbf{x}$ is constant.

Consider the operator $\exp[\tau L_A]\exp[\tau L_B]$, which corresponds to integrating first along the flow induced by Hamiltonian B, followed by that induced by A. We need two results to proceed. The first is the Baker–Campbell–Hausdorff identity,

$$\exp X \exp Y = \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X-Y,[X,Y]] + \langle \text{rest} \rangle\right) \tag{4.71}$$

where $\langle \text{rest} \rangle$ consists of further nested commutators of the operators X and Y. The second is that

$$L_{[A,B]} = -[L_A, L_B], \tag{4.72}$$

which is easily verified using the Jacobi identity. Here $[A, B]$ is the Poisson bracket of the functions A and B, whereas $[L_A, L_B]$ is the commutator of the operators L_A and L_B. Taking these two together, it follows that our simple integrator can be written as

$$\begin{aligned} \exp[\tau L_A] \exp[\tau L_B] &= \exp\left(\tau(L_A + L_B) + \frac{1}{2}\tau^2[L_A, L_B] + \cdots\right) \\ &= \exp\left(\tau(L_H + L_{H_{\text{err}}})\right) \end{aligned} \tag{4.73}$$

in which the error Hamiltonian

$$H_{\text{err}} = \tfrac{1}{2}\tau[A, B] + O(\tau^2). \tag{4.74}$$

Therefore the integrator (4.73) is exact for the surrogate Hamiltonian $H_{\text{surr}} = H + H_{\text{err}}$. Compared to motion under the desired Hamiltonian H, the step $\mathbf{w}(t+\tau) - \mathbf{w}(t)$ given by (4.73) is correct to order $O(\tau)$, making this a first-order integrator.

Notice that swapping the order in which A and B are applied changes the sign of the leading-order term in H_{err}. So, halving the timestep and applying these two integrators in sequence results in the new integrator

$$\left(\exp\left[\tfrac{1}{2}\tau L_A\right] \exp\left[\tfrac{1}{2}\tau L_B\right]\right) \left(\exp\left[\tfrac{1}{2}\tau L_B\right] \exp\left[\tfrac{1}{2} L_A\right]\right). \tag{4.75}$$

Using the identities (4.71) and (4.72) the error Hamiltonian for this new integrator is

$$H_{\text{err}} = \tfrac{1}{12}\tau^2 \left[[A, B], B + \tfrac{1}{2}A\right] + O(\tau^4), \tag{4.76}$$

showing that the leading terms in the error Hamiltonians (4.74) of the back-to-back first-order integrators cancel exactly.

For the case $A = \mathbf{p}^2/2m$ (drift) and $B = V(\mathbf{x})$ (kick) the integrator (4.75) is one of the two forms of the well-known **leapfrog** or **Verlet** integrator (namely the "DKKD" version). Swapping A and B results in the "KDDK" version. For solar-system problems, the Hamiltonian can be written in the form (4.29), which splits into three parts, each of which is easy to integrate in isolation, albeit in very different sets of coordinates.

By chaining together sequences of the form (4.73) or (4.75) with appropriately chosen timesteps τ, it is possible to construct yet higher-order integrators. See Saha & Tremaine (1992).

A disadvantage of symplectic integrators is that the timestep τ cannot be changed adaptively in the same way as is possible in generic integrators. To see this, consider the mapping from $\mathbf{w}(t)$ to $\mathbf{W} = \mathbf{w}(t+\tau) = \mathbf{w}(t) + \Delta\mathbf{w}$, where $\Delta\mathbf{w} = \Delta\mathbf{w}(\mathbf{w}, \tau)$. This is symplectic if $\mathbf{W}$ and $\mathbf{w}$ are related through equation (4.56), namely

$$\left(\frac{\partial \mathbf{W}}{\partial \mathbf{w}}\right) J \left(\frac{\partial \mathbf{W}}{\partial \mathbf{w}}\right)^{\mathrm{T}} = J. \tag{4.77}$$

Making the parameter τ in $\Delta\mathbf{w}$ a function of $\mathbf{w}$ breaks this condition in general.

Chapter 2 of Tremaine (2023) gives a thorough overview of numerical orbit integration schemes – not just symplectic ones. Laskar & Gastineau (2009) present an application to the long-term stability of the solar system.

Exercises

1. Let $(\mathbf{q}, \mathbf{p})$ be canonical coordinates for a system having n degrees of freedom. Consider the transformation $\mathbf{Q} = A\mathbf{q}$, $\mathbf{P} = B\mathbf{p}$ where A and B are $n \times n$ matrices having constant coefficients. How must A and B be related to ensure that $(\mathbf{Q}, \mathbf{P})$ are canonical?

2. Referred to an inertial coordinates $\mathbf{X}$ and momenta $\mathbf{P}$ the Hamiltonian for a system of N planets orbiting a central star is

$$H(\{\mathbf{X}\}, \{\mathbf{P}\}) = \frac{\mathbf{P}_0}{2m_0} + \sum_{n=1}^{N} \frac{\mathbf{P}_n^2}{2m_n} + V, \tag{4.78}$$

where $V(\mathbf{X}_0, \mathbf{X}_1, ..., \mathbf{X}_N)$ accounts for the gravitational interactions among the bodies.
(a) Consider a canonical transformation to new, heliocentric, coordinates $(\mathbf{x}_0, ..., \mathbf{x}_N)$ in which $\mathbf{x}_0 \equiv \mathbf{X}_0$ is the location of the sun and $\mathbf{x}_n \equiv \mathbf{X}_n - \mathbf{X}_0$ are the heliocentric coordinates of the planets ($n = 1, ..., N$). What the the corresponding conjugate momenta $\mathbf{p}_0$ and $\mathbf{p}_1, ..., \mathbf{p}_N$? Show that the Hamiltonian in these new coordinates is given by

$$H(\{\mathbf{x}\}, \{\mathbf{p}\}) = \frac{1}{2m_0}\left(\mathbf{p}_0 - \sum_{n=1}^{N} \mathbf{p}_n\right)^2 + \sum_{n=1}^{N} \frac{\mathbf{p}_n^2}{2m_n} + V. \tag{4.79}$$

(b) consider an alternative transformation in which $\mathbf{x}_1, ..., \mathbf{x}_n$ remain as in part (a), but $\mathbf{x}_0$ is instead defined to be the location of the centre of mass of the whole system. What are the momenta $\mathbf{p}_0$ and $\mathbf{p}_n$ in this case? Show that the Hamiltonian is given by

$$H(\{\mathbf{x}\}, \{\mathbf{p}\}) = \frac{\mathbf{p}_0^2}{2m_{\rm tot}} + \frac{1}{2m_0}\left[\sum_{n=1}^{n} \mathbf{p}_n\right]^2 + \sum_{n=1}^{N} \frac{\mathbf{p}_n^2}{2m_n} + V,$$

where $m_{\rm tot} = m_0 + \sum_{n=1}^{N} m_n$. What advantage, if any, does this coordinate system have over that used in part (a)?

3. Construct a symplectic integator for motion in a rotating frame with Hamiltonian (4.20). [Hint: split the Hamiltonian into $H_{\rm drift} = \frac{1}{2}\mathbf{p}^2 - \boldsymbol{\Omega}_{\rm p} \cdot (\mathbf{r} \times \mathbf{p})$ and $H_{\rm kick} = \Phi(\mathbf{r})$. We can integrate orbits in $H_{\rm kick}$ exactly. Construct an exact integrator for $H_{\rm drift}$.]

4. [BT3.26,3.27] Consider the Hamiltonian $H = A + B$ and let $L_A \bullet \equiv [\bullet, A]$, $L_B \equiv [\bullet, B]$ denote derivatives along the phase flows produced by the functions A and B respectively. Show that the leapfrog integrator

$$\exp\left(\tfrac{1}{2}\tau L_A\right) \exp\left(\tau L_B\right) \exp\left(\tfrac{1}{2}\tau L_A\right)$$

corresponds to motion in a surrogate Hamiltonian $H_{\rm surr} = H + H_{\rm err}$, in which the error Hamiltonian

$$H_{\rm err} = \tfrac{\tau^2}{12}\left[[A, B], B + \tfrac{1}{2}A\right] + O(\tau^4).$$

Consider a new integrator composed of three successive leapfrog steps, the first of length $a\tau$, followed by one of length $b\tau$ and finally another of length $a\tau$, where $2a + b = 1$. Find the constants a and b that kill off all terms lower than $O(\tau^4)$ in the error Hamiltonian of this new integrator.

5. Use invariance of Poisson brackets under canonical transformations to express the collisionless Boltzmann equation (3.8) CBE in spherical polar coordinates (r, ϑ, φ).

Chapter 5

Angle–action coordinates

For regular Hamiltonians we can find a new set of canonical coordinates in which (i) the new momenta are conserved and (ii) the new coordinates increase linearly with time. Coordinates with such friendly properties are attractive candidates to use both for constructing equilibrium models and for investigating how these equilbria respond to perturbations. This chapter outlines how to construct maps between our original $(\mathbf{x}, \mathbf{v})$ coordinates and these new *angle–action* coordinates.

5.1 Generating functions for canonical maps

We have already seen that the mapping obtained by drifting along the flow (4.40) generated by any function $B(\mathbf{q}, \mathbf{p})$ is canonical. This is this most natural way of constructing maps that join smoothly onto the identity map. But this produces only a subset of all canonical maps. For example, in 4.1 we constructed Hamiltonians for a single particle, first using Cartesian (x, y, z) coordinates, then using spherical polars (r, ϑ, φ). Both sets of phase-space coordinates were canonical, but no mapping of the form (4.40) can relate them.

To find a more general way of constructing maps, recall that the condition for a map

$$\begin{aligned} Q_i &= Q_i(\mathbf{q}, \mathbf{p}), \\ P_i &= P_i(\mathbf{q}, \mathbf{p}), \quad (i = 1, \ldots, n), \end{aligned} \tag{5.1}$$

to be canonical is that it preserves the Poincare integral invariants (4.64): that is, for any loops γ we must have

$$\oint_\gamma \mathbf{p} \cdot \mathrm{d}\mathbf{q} = \oint_\gamma \mathbf{P} \cdot \mathrm{d}\mathbf{Q} \tag{5.2}$$

which means that the integrands can differ only by a total differential:

$$\mathbf{p} \cdot \mathrm{d}\mathbf{q} - \mathbf{P} \cdot \mathrm{d}\mathbf{Q} = \mathrm{d}F, \tag{5.3}$$

where F is some function defined on phase space. This F is the **generating function** of the transformation between $(\mathbf{q}, \mathbf{p})$ and $(\mathbf{Q}, \mathbf{P})$. It is an awkward, implicit way of specifying the transformation, but also very powerful: from just one function of $2n$ phase-space coordinates we obtain a complete canonical map.

Type 1 generating functions Let us assume that we can express $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{Q})$. Then we may eliminate $\mathbf{P}$ from F and write $F = F_1(\mathbf{q}, \mathbf{Q})$, a function of both the old and new co-ordinates and time, but not the momenta. Substituting this $F = F_1$ into (5.3) and using the chain rule gives

$$\mathbf{p} \cdot \mathrm{d}\mathbf{q} - \mathbf{P} \cdot \mathrm{d}\mathbf{Q} = \frac{\partial F_1}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} + \frac{\partial F_1}{\partial \mathbf{Q}} \cdot \mathrm{d}\mathbf{Q}. \tag{5.4}$$

As $(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{Q})$ can be varied independently (the equality above has to hold for any loop γ) we must have

$$\mathbf{p} = \frac{\partial F_1}{\partial \mathbf{q}}, \quad \mathbf{P} = -\frac{\partial F_1}{\partial \mathbf{Q}}. \tag{5.5}$$

Thus the function $F_1(\mathbf{q}, \mathbf{Q})$ generates an implicit transformation from $(\mathbf{q}, \mathbf{p}) \to (\mathbf{Q}, \mathbf{P})$. By construction, it satisfies the condition (5.2) and therefore is canonical.

Exercise: What mapping is generated by $F_1 = \mathbf{q} \cdot \mathbf{Q}$? Show that the Hamiltonian $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(\mathbf{p}^2 + \omega^2 \mathbf{q}^2)$ is transformed to $H(\mathbf{Q}, \mathbf{P}) = \frac{1}{2}(\mathbf{Q}^2 + \omega^2 \mathbf{P}^2)$.

Type 2 generating functions Generating functions of the form $F_1(\mathbf{q}, \mathbf{Q})$ are unsuitable for constructing mappings close to the identity. So, instead of writing $F = F_1(\mathbf{q}, \mathbf{Q})$, let us take

$$F = -\mathbf{P} \cdot \mathbf{Q} + F_2(\mathbf{q}, \mathbf{P}), \tag{5.6}$$

in which we treat Q as a function $Q(\mathbf{q}, \mathbf{P})$. Substituting this into (5.3) and using the chain rule to expand $\mathrm{d}F$ gives:

$$\mathbf{p} \cdot \mathrm{d}\mathbf{q} - \mathbf{P} \cdot \mathrm{d}\mathbf{Q} = -\mathbf{Q} \cdot \mathrm{d}\mathbf{P} - \mathbf{P} \cdot \mathrm{d}\mathbf{Q} + \frac{\partial F_2}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} + \frac{\partial F_2}{\partial \mathbf{P}} \cdot \mathrm{d}\mathbf{P}. \tag{5.7}$$

As $(\mathrm{d}\mathbf{P}, \mathrm{d}\mathbf{q})$ vary independently, we must have that

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}. \tag{5.8}$$

This is another implicit canonical mapping between $(\mathbf{q}, \mathbf{p})$ and $(\mathbf{Q}, \mathbf{P})$.

Exercise: Show that $F_2(\mathbf{q}, \mathbf{P}) = \mathbf{q} \cdot \mathbf{P}$ generates the identity map. What mapping does $F_2(\mathbf{q}, \mathbf{P}) = \mathbf{q} \cdot \mathbf{P} + \epsilon \hat{\mathbf{n}} \cdot \mathbf{q}$ produce? What about $F_2 = \mathbf{q} \cdot \mathbf{P} + \epsilon \hat{\mathbf{n}} \cdot \mathbf{P}$? What is the connection between this F_2 and the function $B(\mathbf{z})$ that generates the phase flow (4.40)?

Exercise: Show that when the generating functions F_1 or F_2 depend on time t that the Hamiltonian transforms to

$$K = H + \frac{\partial F}{\partial t}, \tag{5.9}$$

where F is the appropriate GF and the $(\mathbf{q}, \mathbf{p})$ in the original H are expressed as $\mathbf{q} = \mathbf{q}(\mathbf{Q}, \mathbf{P})$ and similarly for $\mathbf{p}$. Thus the canonical maps given by type-1 and type-2 generating functions are given implicitly by

$$\begin{aligned} F_1(\mathbf{q}, \mathbf{Q}): &\quad \mathbf{p} = \frac{\partial F_1}{\partial \mathbf{q}}, \quad \mathbf{P} = -\frac{\partial F_1}{\partial \mathbf{Q}}, \quad K(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial F_1}{\partial t} \\ F_2(\mathbf{q}, \mathbf{P}): &\quad \mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}, \quad K(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial F_2}{\partial t}. \end{aligned} \tag{5.10}$$

5.2 Integrability: the Arnold-Liouville theorem and tori

An integral of motion $I = I(\mathbf{q}, \mathbf{p})$ is a function that commutes with the Hamiltonian: $[I, H] = 0$. Given two integrals of motion, $I_1(\mathbf{q}, \mathbf{p})$ and $I_2(\mathbf{q}, \mathbf{p})$, the Jacobi identity implies that $[[I_1, I_2], H] = 0$. So, $[I_1, I_2]$ is another integral of motion, albeit possibly a trivial one. For example, if L_x and L_y are integrals of motion, then so too is L_z.

If $[I_1, I_2] = 0$ then I_1 and I_2 are said to be **in involution**. That is, their phase flows **commute**. For example, in spherical systems the components of the angular momentum vector $\mathbf{L} = (L_x, L_y, L_z)$ are integrals of motion. The square of the total angular momentum L^2 and any one of (L_x, L_y, L_z) are in involution, but any pair of L_x, L_y and L_z is not.

A Hamiltonian H for a system having n degrees of freedom is **integrable** if it has n independent integrals of motion, $I_1, ..., I_n$, that are in involution with each other, $[I_i, I_j] = 0$. The Hamiltonian itself is always one of these integrals of motion.

Each such integral reduces dimension of accessible phase space by one, since $I_i(\mathbf{q}, \mathbf{p}) = c_i$, where c_i is a constant. So motion in an integrable Hamiltonian is confined to an n-dimensional subvolume M of $2n$-dimensional phase space. The Arnold–Liouville theorem states that, if this n-dimensional subvolume M is compact, then

1. M is can be mapped smoothly onto an n-dimensional torus;

2. we can construct a coordinate system $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ to label points in M, with each θ_i being 2π-periodic and increasing linearly with time, $\dot{\theta}_i = \Omega_i$.

> See §49 of Arnold (1978) for the proof. The important idea is that each integral $I_1, ..., I_n$ defines a flow on M. Starting from any point $\mathbf{x}_0 \in M$ we can reach some other $\mathbf{x} \in M$ by drifting for some time t_1 along the flow generated by I_1, followed by time t_2 drifting along the flow generated by I_2 and so on. Since the flows commute, the order in which we take them to travel from $\mathbf{x}_0$ to $\mathbf{x}$ does not matter. So these $\mathbf{t} = (t_1, ..., t_n)$ can be used to label (at least some) points $\mathbf{x} \in M$. Recalling Liouville's theorem, the torus-like nature of M follows by noticing that the domain of $(t_1, ..., t_n)$ is $\mathbb{R}^n$ whereas M itself is compact by assumption. For certain choices $\{\mathbf{t}_0\}$ of $\mathbf{t}$ we *must* return to $\mathbf{x}_0$. These fixed points $\{\mathbf{t}_0\}$ define a regular lattice on the domain $\mathbb{R}^n$ of $\mathbf{t}$. The angles θ_i and corresponding constant frequencies Ω_i are then constructed by mapping the volume within a fundamental cell of this lattice onto the direct product of n unit circles.

So, these **angle coordinates** $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ label points within the torus. We could label each torus by the values $(I_1, ..., I_n)$ of the n integrals of motion, but it is better to use the Poincare invariants

$$J_i \equiv \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot \mathrm{d}\mathbf{q}, \tag{5.11}$$

where γ_i is any closed loop that involves incrementing θ_i by 2π, with all other θ_j returning to their original values. (A generalisation of Stokes' theorem means that this integral is independent of the precise path taken by γ_i.) There are n such integrals, because any n-dimensional torus admits n independent loops that can't be deformed into one another. The $J_1, ..., J_n$ defined in (5.11) are the **actions** of the torus. The actions are functions $J_i = J_i(I_1, ..., I_n)$ of the original integrals of motion.

The $(\boldsymbol{\theta}, \mathbf{J})$ coordinates are canonical: the mapping $(\mathbf{q}, \mathbf{p}) \leftrightarrow (\boldsymbol{\theta}, \mathbf{J})$ is described by the type-2 generating function (5.6)

$$F_2(\mathbf{q}, \mathbf{J}) = S(\mathbf{q}, \mathbf{J}) = \int_{\mathbf{q}_0}^{\mathbf{q}} \mathbf{p}(\mathbf{q}', \mathbf{J}) \cdot \mathrm{d}\mathbf{q}', \tag{5.12}$$

where $\mathbf{q}_0$ is some arbitrary reference point. It is immediately obvious that $\mathbf{p} = \partial S / \partial \mathbf{q}$. Integrate (5.12) along any loop γ_i along which θ_i increases by 2π with the other θ_j returning to their initial values. The quantity S increases by $\Delta S = \oint_{\gamma_i} \mathbf{p} \cdot \mathrm{d}\mathbf{q} = 2\pi J_i$ using (5.11), showing that $\oint_{\gamma_i} \mathbf{p} \cdot \mathrm{d}\mathbf{q} = \oint_{\gamma_i} \mathbf{J} \cdot \mathrm{d}\boldsymbol{\theta}$ and therefore that J_i is the momentum conjugate to θ_i.

Let's take the value $c_1 = E$ of the Hamiltonian $H(\mathbf{q}, \mathbf{p})$ as the first of the n integrals of motion. Then $S(\mathbf{q}, \mathbf{J})$ must satisfy the **Hamilton–Jacobi equation**,

$$H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}\right) = E, \tag{5.13}$$

in which it is understood that $\mathbf{J}$ is held fixed. Solving this first-order PDE for $S(\mathbf{q}, \mathbf{J})$ allows us to construct mappings to action–angle variables $(\boldsymbol{\theta}, \mathbf{J})$ given a Hamiltonian $H(\mathbf{q}, \mathbf{p})$.

In summary, if a Hamiltonian $H(\mathbf{q}, \mathbf{p})$ is integrable, then we can carry out a canonical map $(\mathbf{q}, \mathbf{p}) \to (\boldsymbol{\theta}, \mathbf{J})$ to new phase-space coordinates in terms of which the Hamiltonian becomes $H = H(\mathbf{J})$ and the equations of motion are simply

$$\dot{\mathbf{J}} = 0, \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}, \tag{5.14}$$

where the vector of frequencies,

$$\boldsymbol{\Omega} \equiv \frac{\partial H}{\partial \mathbf{J}}, \tag{5.15}$$

gives the rate at which the system whizzes around the torus labelled by the actions $\mathbf{J}$. This torus is sometimes called an **invariant torus** because a trajectory that starts on the torus remains on it.

5.3 Angle–action variables for the simple harmonic oscillator

As an example, consider the Hamiltonian $H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. The Hamilton–Jacobi equation (5.13) for $S(q,J)$ in this case is

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 = E, \tag{5.16}$$

which yields

$$\frac{\partial S}{\partial q} = \pm \int \left[2E - \omega^2 q^2\right]^{1/2} \mathrm{d}q. \tag{5.17}$$

Substituting $q = (\sqrt{2E}/\omega)\sin u$ and integrating,

$$S(q,J) = \frac{E}{\omega}(u + \frac{1}{2}\sin 2u) + F(J), \tag{5.18}$$

where $F(J)$ is an arbitrary function of J. The variable u changes by 2π as we move around one complete orbit. The corresponding increase in S is $2\pi E/\omega$. Then by (5.11), the action integral for this orbit is

$$J = \frac{1}{2\pi}\oint \frac{\partial S}{\partial q}\mathrm{d}q = \frac{1}{2\pi}\oint \frac{\partial S}{\partial u}\mathrm{d}u = \frac{E}{\omega}. \tag{5.19}$$

Expressed as a function of J, the Hamiltonian is therefore

$$H(J) = \omega J, \tag{5.20}$$

and, from (5.18), the generating function is

$$S(q,J) = J(u + \frac{1}{2}\sin 2u) + F(J), \tag{5.21}$$

in which $u(q,J)$ is given implicitly by $q = \sqrt{2J/\omega}\sin u$. The only effect of the arbtirary $F(J)$ is to set the "origin" of the angle coordinate $\theta = 0$. Choosing $F = 0$, the angle conjugate to J is given by

$$\begin{aligned}\theta = \frac{\partial S}{\partial J} &= u + \frac{1}{2}\sin 2u + J(1+\cos 2u)\left.\frac{\partial u}{\partial J}\right|_q \\ &= u + \frac{1}{2}\sin 2u - (1+\cos 2u)\frac{\cos u}{\sin u} \\ &= u.\end{aligned} \tag{5.22}$$

Using $p = \partial S/\partial q = J(1+\cos 2u)(\partial u/\partial q)|_J$, the transformation from (θ, J) to (q,p) is

$$q = \sqrt{\frac{2J}{\omega}}\sin\theta, \quad p = \omega\sqrt{\frac{2J}{\omega}}\cos\theta. \tag{5.23}$$

So $(\sqrt{2J},\theta)$ are polar coordinates in a rescaled (q,p) plane.

5.4 Angle–action variables for 2d axisymmetric potentials

We can use the same procedure to construct angle–action variables for any system with two or more degrees of freedom provided we can find a coordinate system in which the Hamilton–Jacobi equation (5.13) separates into

$$\sum_{i=1}^{n} H_i\left(\mathbf{q}_i, \frac{\partial S_i}{\partial q_i}\right) = E, \tag{5.24}$$

with

$$S(\mathbf{q},\mathbf{J}) = \sum_{i=1}^{n} S_i(q_i,\mathbf{J}). \tag{5.25}$$

Unfortunately, most Hamiltonians are not separable, which means that we have to find alternative methods to construct angle–action variables for them. The most important class of Hamiltonian for which the Hamilton–Jacobi separates are those for spherically symmetric potentials. Each orbit in a spherically symmetric potential is confined to its own **orbital plane**, so that the problem reduces to one of motion in a two-dimensional axisymmetric potential. We now focus on this simpler two-dimensional case before turning our attention to the full three-dimensional problem.

We rotate our coordinate system in the Hamiltonian (4.14) so that $\vartheta = \pi/2$ and $p_\vartheta = 0$. The Hamiltonian becomes

$$H(r,p_r,p_\varphi) = \frac{1}{2}\left[p_r^2 + \frac{p_\varphi^2}{r^2}\right] + \Phi(r). \tag{5.26}$$

Writing $S(\mathbf{q},\mathbf{J}) = S_r(r,\mathbf{J}) + S_\varphi(\varphi,\mathbf{J})$, the Hamilton–Jacobi equation is

$$\frac{1}{2}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\varphi}{\partial \varphi}\right)^2\right] + \Phi(r) = E, \tag{5.27}$$

which, separating variables in the usual way, yields the separation constant

$$L_z^2 = \left(\frac{\partial S_\varphi}{\partial \varphi}\right)^2 = p_\varphi^2, \tag{5.28}$$

which is simply the square of the total angular momentum. We choose L_z to have the same sign as p_φ. Around one complete cycle of φ, S_φ increases by $2\pi p_\varphi$, so the **azimuthal action**

$$J_\varphi = L_z = p_\varphi. \tag{5.29}$$

The Hamilton–Jacobi equation becomes

$$\frac{1}{2}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{L_z^2}{r^2}\right] + \Phi(r) = E. \tag{5.30}$$

Integrating r around a loop from pericentre r_- to apocentre r_+ and back to pericentre, the **radial action** is

$$J_r = \frac{1}{2\pi}\oint p_r\,\mathrm{d}r = \frac{1}{2\pi}\oint \frac{\partial S_r}{\partial r}\,\mathrm{d}r = \frac{1}{\pi}\int_{r_-}^{r_+}\left[2(E-\Phi) - \frac{L_z^2}{r^2}\right]^{1/2}\mathrm{d}r, \tag{5.31}$$

because $p_r < 0$ on the return journey from apocentre back to pericentre.

There are few potentials $\Phi(r)$ for which we can use (5.27) to construct an explicit expression for $E = H(J_r, J_\varphi)$. We always have explicit expressions for the frequency vector $\boldsymbol{\Omega} \equiv \partial H/\partial \mathbf{J}$ though. Differentiating (5.45) at constant $L_z = J_\varphi$, noting that the limits r_- and r_+ depend on (E, L_z), but that the integrand vanishes at both endpoints, we have that

$$\frac{1}{\Omega_r} \equiv \left(\frac{\partial J_r}{\partial E}\right)_{L_z} = \frac{1}{\pi}\int_{r_-}^{r_+} \frac{\mathrm{d}r}{\left[2(E-\Phi) - \frac{L_z^2}{r^2}\right]^{1/2}}. \tag{5.32}$$

Notice from (5.32) that $2\pi/\Omega_r$ is simply the radial period T_r of (3.11), as you'd expect. To find Ω_φ we differentiate (5.45) with respect to L_z at constant E to obtain

$$\left(\frac{\partial J_r}{\partial L_z}\right)_E = \frac{L_z}{\pi}\int_{r_-}^{r_+} \frac{\mathrm{d}r}{r^2\left[2(E-\Phi) - \frac{L_z^2}{r^2}\right]^{1/2}}, \tag{5.33}$$

which we recognise as $1/(2\pi)$ times the increment $\Delta\varphi$ in azimuthal angle per radial period, equation (3.12). Then

$$\frac{1}{\Omega_\varphi} = \left(\frac{\partial L_z}{\partial E}\right)_{J_r} = \left(\frac{\partial L_z}{\partial J_r}\right)_E \left(\frac{\partial J_r}{\partial E}\right)_{L_z} = \frac{2\pi}{\Delta\varphi}\frac{1}{\Omega_r}, \tag{5.34}$$

showing that the angular period $T_\varphi = 2\pi/\Omega_\varphi = 2\pi T_r/\Delta\varphi$, in agreement with our earlier derivation in §3.2.

The angles themselves are given by $\theta_i = \partial S/\partial J_i$, where

$$\begin{aligned} S(r,\varphi,\mathbf{J}) = S_r(r,\mathbf{J}) + S_\varphi(\varphi,\mathbf{J}) &= \int^r \frac{\partial S_r(r',\mathbf{J})}{\partial r'}\,\mathrm{d}r' + \int^\varphi \frac{\partial S_\varphi(\varphi',\mathbf{J})}{\partial \varphi'}\,\mathrm{d}\varphi' \\ &= \int^r p_r(r',\mathbf{J})\,\mathrm{d}r' + \int^\varphi p_\varphi(\varphi',\mathbf{J})\,\mathrm{d}\varphi' \\ &= \int_{r_-}^r \pm\left[2(H(J_r,J_\varphi) - \Phi(r')) - \frac{J_\varphi^2}{r'^2}\right]^{1/2}\mathrm{d}r' + \int_0^\varphi J_\varphi\,\mathrm{d}\varphi'. \end{aligned} \tag{5.35}$$

In the last line we have chosen $\theta_r = 0$ to correspond to pericentre, with $\theta_\varphi = 0$ at $\varphi = 0$. The positive square root in the first integral is used when r is increasing (i.e., $0 < \theta_r < \pi$), the negative sign when it is decreasing ($\pi < \theta_r < 2\pi$). So,

$$\theta_r = \frac{\partial S}{\partial J_r} = \frac{\partial H}{\partial J_r}\int_{r_-}^r \pm\frac{\mathrm{d}r'}{\left[2(H-\Phi(r')) - J_\varphi^2/r'^2\right]^{1/2}}, \tag{5.36}$$

with a similar expression for $\theta_\varphi(r,\varphi,\mathbf{J})$. Differentiating this expression with respect to time, recognizing that the denominator of the integrand is $\dot r$, we have $\dot\theta_r = \partial H/\partial J_r = \Omega_r$ as expected. Just to spell it out,

$$\theta_\varphi = \frac{\partial S}{\partial J_\varphi} = \frac{\partial H}{\partial J_\varphi}\int_{r_-}^r \pm\frac{\mathrm{d}r'}{\left[2(H-\Phi(r')) - J_\varphi^2/r'^2\right]^{1/2}} - J_\varphi\int_{r_-}^r \pm\frac{\mathrm{d}r'}{r'^2\left[2(H-\Phi(r')) - J_\varphi^2/r'^2\right]^{1/2}} + \varphi. \tag{5.37}$$

The time derivative of the second integral is $-J_\varphi/r^2 = -\dot\varphi$, which cancels out the $\dot\varphi$ from the last term.

5.5 Angle–action variables for spherical potentials

The procedure for the three-dimensional case is only slightly more involved. The Hamiltonian (4.14),

$$H(r,\vartheta,p_r,p_\vartheta,p_\varphi) = \frac{1}{2}\left[p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\vartheta}\right] + \Phi(r), \tag{5.38}$$

has three independent integrals of motion in involution: H, p_φ and $p_\varphi^2 + p_\vartheta^2/\sin^2\vartheta$. Writing $S(\mathbf{q},\mathbf{J}) = S_r(r,\mathbf{J}) + S_\vartheta(\vartheta,\mathbf{J}) + S_\varphi(\varphi,\mathbf{J})$, the Hamilton–Jacobi equation is

$$\frac{1}{2}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\vartheta}{\partial\vartheta}\right)^2 + \frac{1}{r^2\sin^2\vartheta}\left(\frac{\partial S_\varphi}{\partial\varphi}\right)^2\right] + \Phi(r) = E, \tag{5.39}$$

which, separating variables in the usual way, yields the pair of separation constants

$$\begin{aligned} L_z^2 &= \left(\frac{\partial S_\varphi}{\partial\varphi}\right)^2 = p_\varphi^2, \\ L^2 &= \left(\frac{\partial S_\vartheta}{\partial\vartheta}\right)^2 + \frac{L_z^2}{\sin^2\vartheta} = p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}, \end{aligned} \tag{5.40}$$

which are just the z component of angular momentum and the square of the total angular momentum. These are related to E by

$$\frac{1}{2}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{L^2}{r^2}\right] + \Phi(r) = E. \tag{5.41}$$

We choose L_z to have the same sign as p_φ, and L to be positive.

Around one complete cycle of φ, S_φ increases by $2\pi p_\varphi$, so the **azimuthal action** is, as before,

$$J_\varphi = L_z = p_\varphi. \tag{5.42}$$

The variables (ϑ, p_ϑ) circulate around the loop given by $L^2 = p_\vartheta^2 + L_z^2/\sin^2\vartheta$, in which ϑ is limited to the region $\sin\vartheta > |L_z|/L \equiv \sin\vartheta_{\min}$. Over the course of one cycle S_ϑ increases by

$$\begin{aligned}\Delta S_\vartheta &= 4\int_{\vartheta_{\min}}^{\pi/2} \frac{\partial S_\vartheta}{\partial\vartheta}\mathrm{d}\vartheta = 4\int_{\vartheta_{\min}}^{\pi/2}\left[L^2 - \frac{L_z^2}{\sin^2\vartheta}\right]^{1/2}\mathrm{d}\vartheta \\ &= 2\pi(L - |L_z|),\end{aligned} \tag{5.43}$$

and the **latitudinal action** is therefore

$$J_\vartheta = L - |L_z|. \tag{5.44}$$

The **radial action** is again

$$J_r = \frac{1}{\pi}\int_{r_-}^{r_+}\left[2(E-\Phi) - \frac{L^2}{r^2}\right]^{1/2}\mathrm{d}r, \tag{5.45}$$

where r_- and r_+ are the orbit's peri- and apo-centre radii respectively.

The rotational symmetry of the Hamiltonian means that it can depend on the actions J_ϑ and J_φ only in the combination $L = J_\vartheta + |J_\varphi|$. Therefore

$$H = H(J_r, J_\vartheta + |J_\varphi|), \tag{5.46}$$

and the magnitudes of the azimuthal and latitudinal frequencies are equal, $|\Omega_\vartheta| = |\Omega_\varphi|$. So, the quantity $\theta_\vartheta - \mathrm{sgn}(J_\varphi)\theta_\varphi$ an integral of motion in addition to $(J_r, J_\vartheta, J_\varphi)$.

Angle-action coordinates are not unique, however: we can always construct new angle-action variables $(\boldsymbol{\theta}', \mathbf{J}')$ by taking appropriate linear combinations of the old ones $(\boldsymbol{\theta}, \mathbf{J})$, as follows (Morbidelli, 2002, §1.9.1). Let A be a 3×3 matrix with integer coefficients having unit determinant. Then the map

$$\mathbf{J}' = (A^{-1})^{\mathrm{T}}\mathbf{J}, \quad \boldsymbol{\theta}' = A\boldsymbol{\theta}, \tag{5.47}$$

to new coordinates $(\boldsymbol{\theta}', \mathbf{J}')$ is canonical (exercise: why?). Moreover, increasing any one of the θ_i' by 2π defines a distinct closed loop on the torus, so the new $(\boldsymbol{\theta}', \mathbf{J}')$ are angle–action coordinates.

Let us use this idea to promote $L = J_\vartheta + |J_\varphi|$ to the status of an action. Take

$$\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix} = \begin{pmatrix}0&0&1\\0&1&\mathrm{sgn}(J_\varphi)\\1&0&0\end{pmatrix}\begin{pmatrix}J_r\\J_\vartheta\\J_\varphi\end{pmatrix}, \quad \begin{pmatrix}\theta_r\\\theta_\vartheta\\\theta_\varphi\end{pmatrix} = \begin{pmatrix}0&0&1\\0&1&0\\1&\mathrm{sgn}(J_\varphi)&0\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}, \tag{5.48}$$

the matrix in the first inequality being $(A^{-1})^{\mathrm{T}}$, the second its transpose A^{-1}. That is,

$$\begin{aligned}\theta_1 = \theta_\varphi - \mathrm{sgn}(J_\varphi)\theta_\vartheta, &\quad J_1 = J_\varphi = L_z,\\ \theta_2 = \theta_\varphi, &\quad J_2 = J_\vartheta + |J_\varphi| = L,\\ \theta_3 = \theta_r, &\quad J_3 = J_r.\end{aligned} \tag{5.49}$$

In these new coordinates $H = H(J_2, J_3)$, so that the angle θ_1 becomes a fourth integral of motion in addition to (J_1, J_2, J_3).

To understand the geometrical meaning of these coordinates, recall that in a spherical potential the orbit is confined to a plane whose normal is set by the direction of the angular momentum vector $\mathbf{L}$. Following §2.3, let (x', y') be Cartesian coordinates in this plane. Changing the orientation of $\mathbf{L}$ away from $(0, 0, 1)$ changes the particle's coordinates to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix}, \tag{5.50}$$

the inclination angle i and longitude of the ascending node Ω set by the direction of $\mathbf{L}$. From this it is clear that the magnitude of $\cos i$ gives the magnitude of the normal to the orbital plane $\mathbf{L}$ onto the z-axis, with the sign chosen so that $J_\varphi > 0$ when $i < \frac{\pi}{2}$. Therefore

$$\cos i = \frac{J_1}{J_2}. \tag{5.51}$$

By considering the Hamilton–Jacobi equation expressed in these new coordinates, the angles θ_i $(i = 1, 2, 3)$ are given by $\theta_i = \partial S/\partial J_i$, where

$$\begin{aligned} S(r, \vartheta, \varphi, J_1, J_2, J_3) &= \int p_r(r', \mathbf{J})\,\mathrm{d}r' + \int p_\vartheta(\vartheta', \mathbf{J})\,\mathrm{d}\vartheta' + \int p_\varphi(\varphi', \mathbf{J})\,\mathrm{d}\varphi' \\ &= \int_{r_-}^{r} s_r \left[2(H(J_2, J_3) - \Phi(r')) - \frac{J_2^2}{r'^2}\right]^{1/2} \mathrm{d}r' + \int_{\pi/2}^{\vartheta} s_\vartheta \left[J_2^2 - \frac{J_1^2}{\sin^2\vartheta'}\right]^{1/2} \mathrm{d}\vartheta' + J_1\varphi, \end{aligned} \tag{5.52}$$

with the signs $s_\vartheta = \pm 1$, $s_r = \pm 1$ chosen to ensure that the integrals increase monotonically along the orbit.[1] Consider the orbit just after it has passed upwards through the plane $z = 0$, so that $\dot{z} > 0$ and $\dot{\vartheta} < 0$. Then $p_\vartheta < 0$, meaning $s_\vartheta = -1$. Differentiating (5.52) and using (5.51)

$$\theta_1 = \frac{\partial S}{\partial J_1} = \operatorname{sgn}(J_1) \int_{\pi/2}^{\vartheta} \frac{\mathrm{d}\vartheta'}{\sin\vartheta' \sqrt{\sin^2\vartheta'/\cos^2 i - 1}} + \varphi. \tag{5.53}$$

To perform the integral, consider

$$\mathrm{d}\left[\sin^{-1}(\cot i \cot\vartheta)\right] = -\frac{\cot i}{\sin^2\vartheta} \frac{\mathrm{d}\vartheta}{\sqrt{1 - \cos^2 i \cot^2\vartheta}} = -\frac{\operatorname{sgn}(\cos i)\,\mathrm{d}\vartheta}{\sin\vartheta\sqrt{(1 + \tan^2 i)\sin^2\vartheta - \cos^2\vartheta}} \tag{5.54}$$

$$= -\operatorname{sgn}(\cos i) \frac{\mathrm{d}\vartheta}{\sin\vartheta\sqrt{\sin^2\vartheta/\cos^2 i - 1}}. \tag{5.55}$$

But $\operatorname{sgn}(J_1) = \operatorname{sgn}(\cos i)$ and so equation (5.53) becomes $\theta_1 = -u + \varphi$, where u is given by $\sin u = \cot i \cot\vartheta$. In §2 we saw (2.27) that this $\sin u = \cot i \cot\vartheta = \sin(\varphi - \Omega)$. Considering how u and φ vary over the course of an orbit, it follows that $u = \varphi - \Omega$. Therefore

$$\theta_1 = \Omega, \tag{5.56}$$

the longitude of the ascending node. This is the fourth integral of motion in a general spherically symmetric potential.

5.6 Angle–action variables for the Kepler Hamiltonian: Delaunay elements

Taking $\Phi(r) = -GM/r$ in (5.38) gives the Kepler Hamiltonian. In this case radial action can be calculated analytically. It is

$$J_r = \frac{1}{\pi} \int_{r_-}^{r_+} \sqrt{2\left(E + \frac{GM}{r}\right) - \frac{L^2}{r^2}}\,\mathrm{d}r = \frac{GM}{\sqrt{-2E}} - L. \tag{5.57}$$

[1] In the integral for each of (r, ϑ, φ) we are free to add an arbitrary constant function of the other two coordinates. The choice here, set by the lower bounds on the integrals, corresponds to the convention that $\boldsymbol{\theta} = \mathbf{0}$ at $(r, \vartheta, \varphi) = (r_-, \frac{\pi}{2}, 0)$.

Therefore the Kepler Hamiltonian is a very simple function of the actions, namely

$$H = -\frac{(GM)^2}{2(J_r + J_\vartheta + |J_\varphi|)^2} = -\frac{(GM)^2}{2(J_r + L)^2}. \tag{5.58}$$

Following (5.48), this becomes the even simpler

$$H = -\frac{(GM)^2}{2J_3^2}, \tag{5.59}$$

when we use (5.47) to define a third set of angle–action variables, $(\theta_1, \theta_2, \theta_3, J_1, J_2, J_3)$ through

$$\begin{aligned} &\underbrace{J_1 = J_\varphi = L_z, \quad \theta_1 = \theta_\varphi - \mathrm{sgn}(J_\varphi)\theta_\vartheta,}_{(H,h)} \\ &\underbrace{J_2 = J_\vartheta + |J_\varphi| = L, \quad \theta_2 = \theta_\vartheta - \theta_r,}_{(G,g)} \\ &\underbrace{J_3 = J_r + L, \quad \theta_3 = \theta_r.}_{(L,l)} \end{aligned} \tag{5.60}$$

These new angle–action coordinates are the **Delaunay elements** of the orbit: the underbraces in the equation above give the more usual labels for them. Notice that the frequencies are $\Omega_1 = \Omega_2 = 0$, $\Omega_3 = (GM)^2/J_3^3$.

There is a simple relationship between these Delaunay elements and the orbital elements introduced in §2.3. Clearly $J_3^2 = GMa$ and $J_2^2 = GMa(1-e^2)$, so that $J_1^2 = GMa(1-e^2)\cos^2 i$. To understand the angles, apply $\theta_i = \partial S/\partial J_i$ to to the Hamilton–Jacobi generating function (5.52). We have already seen (equ 5.56) that $\theta_1 = \Omega$, the longitude of the ascending node. Making the substitution $x = \cos\vartheta/\sin i$ to carry out the ϑ integral in (5.52) gives

$$\theta_2 = \frac{\partial S}{\partial J_2} = -\int_{r_-}^{r} s_r \frac{L\,\mathrm{d}r'}{r'^2\left[2(H(J_2,J_3) - \Phi(r')) - \frac{L^2}{r'^2}\right]^{1/2}} + \sin^{-1}\left(\frac{\cos\vartheta}{\sin i}\right), \tag{5.61}$$

bearing in mind that the value of the arcsine in the second term must be chosen so that θ_2 varies smoothly. The integrand of the first term is $L\mathrm{d}r'/r'^2 v_r = L\,dt/r'^2 = \mathrm{d}\psi$, where ψ is the angle in the orbital plane measured from the origin between the particle's present position and its pericentre.[2] Therefore the first term is simply $-\psi$. Substituting $(x', y') = r(\cos\psi, \sin\psi)$ into (2.20) and looking at the $z = r\cos\vartheta$ component of the result shows that $\cos\vartheta/\sin i = \sin(\omega + \psi)$. So, the second term is $\omega + \psi$. Adding both contributions we have that $\theta_2 = \omega$, the argument of pericentre.

The remaining angle,

$$\theta_3 = \frac{\partial S}{\partial J_3} = \frac{\partial H}{\partial J_3}\int_{r_-}^{r} s_r \frac{\mathrm{d}r'}{\left[2(H(J_2,J_3) - \Phi(r')) - \frac{L^2}{r'^2}\right]^{1/2}}, \tag{5.62}$$

the integrand of which is simply $\Omega_r \mathrm{d}r/v_r = \Omega_r \mathrm{d}t$. Therefore $\theta_3 = w$, the mean anomaly.

In summary, we've shown that $H = -(GM)^2/2J_3^2$ with

$$\begin{aligned} &\underbrace{J_1 = \sqrt{GMa(1-e^2)}\cos i, \quad \theta_1 = \theta_\varphi - \mathrm{sgn}(J_\varphi)\theta_\varphi = \Omega,}_{(H,h)} \\ &\underbrace{J_2 = \sqrt{GMa(1-e^2)}, \quad \theta_2 = \theta_\vartheta - \theta_r = \omega,}_{(G,g)} \\ &\underbrace{J_3 = \sqrt{GMa}, \quad \theta_3 = \theta_r = w.}_{(L,l)} \end{aligned} \tag{5.63}$$

[2] In the two-dimensional nursery of §2.2 we called this angle φ.

A problem with these Delaunay variables is that the angles (Ω, ω, w) are not well defined whenever $i = 0$ or $e = 0$. To remedy this, let us introduce **modified Delaunay variables** via

$$\begin{aligned} \underbrace{J_\varpi = \sqrt{GMa}(1-\sqrt{1-e^2}), \quad \theta_\varpi = -(\Omega+\omega),}_{(P,p)} \\ \underbrace{J_\Omega = \sqrt{GMa(1-e^2)}(1-\cos i), \quad \theta_\Omega = -\Omega,}_{(Q,q)} \\ \underbrace{J_\lambda = \sqrt{GMa}, \quad \theta_\lambda = \lambda \equiv \Omega+\omega+w,}_{(\Lambda,\lambda)} \end{aligned} \tag{5.64}$$

in terms of which $H = -(GM)^2/2J_\lambda^2$. Here we have introduced two new angles,

$$\begin{aligned} \text{(longitude of pericentre)} \quad \varpi &\equiv \omega + \Omega, \\ \text{(mean longitude)} \quad \lambda &\equiv \Omega + \omega + w. \end{aligned} \tag{5.65}$$

λ is always well defined, whereas ϖ and Ω are undefined when the corresponding action is zero: these are polar coordinates. Notice that

$$\begin{aligned} J_\varpi &\propto e^2, \quad e \ll 1, \\ J_\Omega &\propto i^2, \quad i \ll 1. \end{aligned} \tag{5.66}$$

5.7 Other potentials with explicit expressions for actions

5.7.1 Isochrone

The isochrone potential is

$$\Phi(r) = -\frac{GM}{b+\sqrt{b^2+r^2}}. \tag{5.67}$$

In the limit $b \to 0$ this becomes the Kepler potential, whereas in the limit $b \to \infty$ it becomes the spherical simple harmonic oscillator. The isochrone is important because it is the most general poetntial for which all angle–action variables can be obtained analytically from the ordinary phase-pace coordinates $(\mathbf{x}, \mathbf{p})$. See §3.5 of Binney & Tremaine (2008) for details.

Why the name? In this potential the radial period of an orbit having energy E per unit mass is $T_r = 2\pi GM/(-2E)^{3/2}$, which is independent of angular momentum.

5.7.2 Stäckel potentials

The German mathematician Paul Stäckel showed that only coordinate system in which the Hamiltonian–Jacobi equation $H = \frac{1}{2}\mathbf{p}^2 + \Phi(\mathbf{x}) = E$ separates is confocal ellipsoidal coordinates, of which Cartesian, spherical polar and cylindrical polars are just limiting cases. Confocal spheroidal coordinates (u, v, φ) are the axisymmetric version. They are related to cylindrical coordinates via

$$\begin{aligned} R &= \Delta \sinh u \sin v, \\ z &= \Delta \cosh u \cos v. \end{aligned} \tag{5.68}$$

If the potential is of the form

$$\Phi(u, v) = \frac{U(u) - V(v)}{\sinh^2 u + \sin^2 v}, \tag{5.69}$$

for some functions $U(u)$ and $V(v)$, then it is easy to show that the Hamilton–Jacobi equation separates, providing two action integrals in addtion to $J_\varphi = L_z$ (e.g., §3.5 of Binney & Tremaine, 2008). de Zeeuw (1985) is the canonical reference for the properties of these **Stäckel potentials**, including the case of triaxial symmetry.

5.8 Numerical construction of angle–action variables

Numerical experiments show that in most realistic galaxy Hamiltonians most orbits are regular: i.e., orbits have as many isolating integrals of motion as the degrees of freedom (2 or 3) of the system. How to show this? For systems with two degrees of freedom (e.g., axisymmetric potentials) this is most easily shown simply by plotting surfaces of section, which reduce four-dimensional phase space to a sequence of two-dimensional slices. Each surface of section is a slice through the orbits' phase-space tori.

This does not work in systems with three degrees of freedom. Instead one can use the fact that regular orbits are quasiperiodic, whereas irregular orbits are not. Fourier analysis of the $\mathbf{x}(t)$ time series of numerically integrated orbits is the natural way to identify their fundamental frequencies (if any) and therefore to test for regularity. The details are subtle though: see §3.7.3 of Binney & Tremaine (2008).

How to construct angle–action variables for these more general Hamiltonians $H(\mathbf{x},\mathbf{p})$? One way is to start from a "toy" Hamiltonian $H'(\mathbf{J}',\boldsymbol{\theta}')$ for which we can map easily between $(\mathbf{x},\mathbf{p})$ and $(\boldsymbol{\theta}',\mathbf{J}')$. An example of such a toy Hamiltonian is that of the isochrone model in §5.7.1. Having chosen H', we construct canonical map between these toy coordinates $(\boldsymbol{\theta}',\mathbf{J}')$ and the angle–action coordinates $(\boldsymbol{\theta},\mathbf{J})$ of the target H as follows. Exploit the periodicity of the angle variables and represent the map by a type-2 generating function

$$S(\boldsymbol{\theta}',\mathbf{J}) = \boldsymbol{\theta}'\cdot\mathbf{J} + \sum_{\mathbf{n}} S_{\mathbf{n}}(\mathbf{J})\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}'}, \tag{5.70}$$

so that

$$\begin{aligned}\mathbf{J}' &= \frac{\partial S}{\partial\boldsymbol{\theta}'} = \mathbf{J} + \sum_{\mathbf{n}} \mathrm{i}\mathbf{n} S_{\mathbf{n}}(\mathbf{J})\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}'},\\ \boldsymbol{\theta} &= \frac{\partial S}{\partial\mathbf{J}} = \boldsymbol{\theta}' + \sum_{\mathbf{n}} \frac{\partial S_{\mathbf{n}}}{\partial\mathbf{J}}(\mathbf{J})\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}'}.\end{aligned} \tag{5.71}$$

Note that $S_{\mathbf{n}}(\mathbf{J}) = S^{\star}_{-\mathbf{n}}(\mathbf{J})$ if S is to be real.

Here is one way of choosing the coefficients $S_{\mathbf{n}}(\mathbf{J})$ that specify this generating function. We start by choosing a value of $\mathbf{J}$ and setting up a regular grid in toy angles $\boldsymbol{\theta}'$. For a given choice of $\mathbf{S}_n$ then the first of equations (5.71) gives the corresponding toy actions. We can use the toy Hamiltonian H_0 to turn these $(\boldsymbol{\theta}',\mathbf{J}')$ into values of $(\mathbf{q},\mathbf{p})$ and then to values of our target $H(\mathbf{q},\mathbf{p})$. By adjusting the coefficents $S_{\mathbf{n}}$ to minimise the spread in the values of $H(\mathbf{q},\mathbf{p})$ we construct a mapping that lies on the torus of action $\mathbf{J}$ generated by H.

An alternative is is to use numerical orbit integration in the target H. This sets $\mathbf{J}$, albeit implicitly. We can use orbit integration to produce time series $(\mathbf{q}(t),\,\mathbf{p}(t))$ and then map these into angle-action variables $(\boldsymbol{\theta}'(t),\mathbf{J}'(t))$ for the toy Hamiltonian H'. Then, writing $\boldsymbol{\theta} = \boldsymbol{\Omega}t$, equation (5.71) becomes a set of linear simultaneous equations for $\boldsymbol{\Omega}$, $\mathbf{J}$, $S_{\mathbf{n}}(\mathbf{J})$ and their derivatives $\partial S_{\mathbf{n}}/\partial\mathbf{J}$.

A third way is to average the first of equs (5.71) over toy angles $\boldsymbol{\theta}'$, obtaining $\mathbf{J} = \frac{1}{(2\pi)^3}\int\mathbf{J}'\,\mathrm{d}\boldsymbol{\theta}'$. The problem reduces to carrying out this integral numerically given irregularly sampled points in $\boldsymbol{\theta}'$.

Here we've just touched on the general idea, but success depends on details such as the choice of toy Hamiltonian H' and limiting $|\mathbf{n}|$. Sanders & Binney (2016) review these and other action-estimation methods in galactic dynamics.

5.9 Resonances and degeneracy

Apart from some special situations, the frequencies $\boldsymbol{\Omega}$ are in general **incommensurable**: that is, the only vector of integers $\mathbf{k}\equiv(k_1,...,k_n)$ for which $\mathbf{k}\cdot\boldsymbol{\Omega}=0$ is the trivial $\mathbf{k}=0$. The motion is then **quasi-periodic** or **non-resonant**. Given any function $f(\boldsymbol{\theta})$ on the torus, it follows then by expanding $f(\boldsymbol{\theta}) = \sum_{\mathbf{k}} f_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}$ that the mean value of f averaged over the torus is

$$\langle f\rangle_{\boldsymbol{\theta}} = \frac{1}{(2\pi)^n}\int f(\boldsymbol{\theta})\mathrm{d}^n\boldsymbol{\theta} = \frac{1}{(2\pi)^n}\sum_{\mathbf{k}} f_{\mathbf{k}}\int\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\mathrm{d}^n\boldsymbol{\theta} = f_0, \tag{5.72}$$

where f_0 is the $\mathbf{k} = 0$ component in the Fourier expansion of $f(\boldsymbol{\theta})$. On the other hand, using $\boldsymbol{\theta}(t) = \boldsymbol{\Omega}t + \boldsymbol{\theta}_0$ the time averaged value of the function $f(\boldsymbol{\theta}(t))$ is

$$\begin{aligned}\langle f\rangle_T &\equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T f(\boldsymbol{\theta}(t))\mathrm{d}t = \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{\mathbf{k}} f_{\mathbf{k}}\mathrm{e}^{\mathrm{i}(\boldsymbol{\Omega}t+\boldsymbol{\theta}_0)}\mathrm{d}t \\ &= f_0 + \lim_{T\to\infty}\frac{1}{T}\sum_{\mathbf{k}\neq 0} f_{\mathbf{k}}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{\Omega}T}-1}{\mathrm{i}\boldsymbol{\Omega}\cdot\mathbf{k}} = f_0,\end{aligned} \tag{5.73}$$

provided there is no $\mathbf{k}$ for which $\mathbf{k}\cdot\boldsymbol{\Omega} \neq 0$. That is, the time average of a function defined on a nonresonant torus is equal to its angle average. In particular, the length of time that an orbit spends within an chunk V of the torus is proportional to the volume occupied by V. A consequence of this is that any single orbit fills the torus densely: starting from any $\boldsymbol{\theta}_0$, we come arbitrarily close to any other $\boldsymbol{\theta}$ in the limit $t \to \infty$.

Exercises

1. (a) Sketch the (x, p) phase plane for the pendulum Hamiltonian

$$H = \tfrac{1}{2}p^2 - \omega_0^2\cos x \tag{5.74}$$

and identify three different orbit families.

(b) Let $h = \sqrt{\frac{1}{2}\left(1+\frac{H}{\omega_0^2}\right)}$. Show that orbits with $0 \le h < 1$ have actions given by

$$J = \frac{8\omega_0}{\pi}\left[E(h) - (1-h^2)K(h)\right] \tag{5.75}$$

and frequencies

$$\Omega = \frac{\pi\omega_0}{2K(h)}, \tag{5.76}$$

where

$$\begin{aligned}K(k) &= \int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \\ E(k) &= \int_0^{\pi/2}\mathrm{d}\theta\,\sqrt{1-k^2\sin^2\theta}\end{aligned} \tag{5.77}$$

are the complete elliptic integrals of the first and second kinds, respectively. [Hint: substitute $\sin\frac{1}{2}x = h\sin\theta$.]

(c) Obtain expressions for J and Ω when $h > 1$. Summarise your results by explaining how to calculate actions and frequencies for each of the three orbit families identified in (a).

(d) We can think of the pendulum as modelling a free particle having Hamiltonian $H_0 = \frac{1}{2}p^2$ that is subject to a perturbation $\epsilon H_1(x) = -\omega_0^2\cos x$. The perturbation introduces a qualitative change in the nature of the orbits within some area of the phase plane. How does this area scale with the strength ϵ of the perturbation?

2. This question, based on Exercise 3.34 of Binney & Tremaine (2008), is intended as a (mostly) straightforward demonstration of the construction of actions for Stäckel potentials (5.69).

(a) Sketch contours of constant u and contours of constant v in the (R, z) plane. What limits should we place on the range of u and v to ensure that each point of the plane is covered just once?

(b) Obtain expressions for the momenta p_u and p_v conjugate to (u, v) and show that each may be expressed as $\Delta^2(\sinh^2 u + \sin^2 v)$ times a generalised velocity, $\dot{u}$ or $\dot{v}$. Construct the Hamiltonian $H(u, v, p_u, p_v, p_\varphi)$.

(c) Assuming a generating function $S(u, v, \mathbf{J}) = S_u(u, \mathbf{J}) + S_v(v, \mathbf{J}) + S_\varphi(\varphi, \mathbf{J})$, how are p_u and p_v related to S_u and S_v? Separate the Hamilton–Jacobi equation $H(u, v, p_u, p_v, p_\varphi) = E$ to show that

$$\begin{aligned} I &= [E \sinh^2 u - U(u)] - \frac{1}{2\Delta^2}\left[p_u^2 + \frac{p_\varphi^2}{\sinh^2 u}\right] \\ &= -[E \sin^2 v + V(v)] + \frac{1}{2\Delta^2}\left[p_v^2 + \frac{p_\varphi^2}{\sin^2 v}\right] \end{aligned} \tag{5.78}$$

is an integral of motion.

(d) By eliminating E from this pair of expressions show that

$$I = \frac{(p_v^2 - 2\Delta^2 V)\sinh^2 u - (p_u^2 + 2\Delta^2 U)\sin^2 v}{2\Delta^2(\sinh^2 u + \sin^2 v)} + \frac{p_\varphi^2}{2\Delta^2 \sinh^2 u \sin^2 v}. \tag{5.79}$$

Show also that $[f(a, b), A] = \frac{\partial f}{\partial a}[a, A] + \frac{\partial f}{\partial b}[b, A]$ for any sufficiently smooth functions a, b, $f(a, b)$ and A and, hence or otherwise, that $[I, H] = 0$.

(e) Write down an expression for each the actions J_u and J_v as an integral over a functions that involves E, I and either $U(u)$ and $V(v)$. Hence show that J_u and J_v are involution, $[J_u, J_v] = 0$.

Chapter 6

Perturbation theory: orbits

Suppose that we have constructed angle–action variables for the Hamiltonian H_0. As we have seen, Hamilton's equations are then simply

$$\dot{\mathbf{J}} = 0, \qquad \dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}(\mathbf{J}), \tag{6.1}$$

where the angles $\boldsymbol{\theta}$ increase at the rate $\boldsymbol{\Omega}(\mathbf{J}) = \partial H_0/\partial \mathbf{J}$.

Now add a perturbation ϵH_1 to our original H_0, where ϵ is a small parameter. Our mastery of angle–action variables means that we can express $H_1 = H_1(\boldsymbol{\theta}, \mathbf{J})$. Hamilton's equations for this perturbed system are

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}(\mathbf{J}) + \epsilon \mathbf{f}(\boldsymbol{\theta}, \mathbf{J}), \quad \dot{\mathbf{J}} = \epsilon \mathbf{g}(\boldsymbol{\theta}, \mathbf{J}), \tag{6.2}$$

with $\mathbf{f} = \partial H_1/\partial \mathbf{J}$ and $\mathbf{g} = -\partial H_1/\partial \boldsymbol{\theta}$. Our mastery does not extend, however, to constructing angle–action variables for the new Hamiltonian $H_0 + \epsilon H_1$, which depends on $\boldsymbol{\theta}$ as well as $\mathbf{J}$. On the other hand, we are usually more interested in the long-term behaviour of the system (e.g., in how the semimajor axes and eccentricities of the planets evolve) than in the details of how the rapidly varying angles $\boldsymbol{\theta}$ change. It turns out that the **averaged** system,

$$\dot{\mathbf{J}} = \epsilon \bar{\mathbf{g}}(\mathbf{J}), \tag{6.3}$$

where

$$\bar{g}(\mathbf{J}) \equiv \frac{1}{(2\pi)^n} \int g(\boldsymbol{\theta}, \mathbf{J}) \mathrm{d}^n \boldsymbol{\theta}, \tag{6.4}$$

is often a good approximation to the evolution produced by the original equations (6.2). This is an ancient idea: when studying perturbations of planets, Gauss proposed to distribute the mass of each planet proportionally to the fraction of time it spent in each segment of the orbit. This reduces the solar system to a system of interacting massive rings.

This averaging principle can be couched in more formal language (see §6.5 later), but for now let's just assume that it works. To verify that there are cases for which it really does work, consider the toy example of a one-dimensional Hamiltonian $H_0 = H_0(J)$ for which $\Omega = \mathrm{d}H/\mathrm{d}J \neq 0$. Apply a perturbation $\epsilon g(\theta)$, which for simplicity we take to be independent of J. Then the full solution to Hamilton's equation $\dot{J} = \epsilon g$ is

$$\begin{aligned} J(t) - J(0) &= \epsilon \int_0^t g(\theta_0 + \Omega t') \mathrm{d}t' \\ &= \epsilon \bar{g} t + \epsilon \int_0^t [g(\theta_0 + \Omega t') - \bar{g}] \mathrm{d}t', \end{aligned} \tag{6.5}$$

where θ_0 is the initial angle. The second term in the RHS is a periodic function. Therefore it is bounded and the evolution of $J(t)$ consists of small oscillations (second term) superimposed on the long-term growth, $\epsilon \bar{g} t$ (first term). The averaged equation of motion (6.3) for this system is $\dot{J} = \epsilon \bar{g}$, which, although it ignores the oscillations on timescales $\lesssim \Omega^{-1}$, correctly captures the longer-term, **secular** evolution over many orbital periods.

We have not yet used the fact that the perturbation is (usually) Hamiltonian, so that $\mathbf{g} = -\partial H_1/\partial\boldsymbol{\theta}$. By the periodicity of H_1 in the angles, we can write $H_1 = \sum_{\mathbf{k}} S_{\mathbf{k}}(\mathbf{J})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\theta}$ for some $S_{\mathbf{k}}(\mathbf{J})$. Therefore $\mathbf{g}(\boldsymbol{\theta},\mathbf{J}) = -\mathrm{i}\sum_{\mathbf{k}} S_{\mathbf{k}}(\mathbf{J})\mathbf{k}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}$ and so $\bar{\mathbf{g}} = 0$: there should be no secular evolution in Hamiltonian systems! But this argument relies on the assumption that the averaging principle is approximately correct. This is not true close to resonances, $\mathbf{k}\cdot\boldsymbol{\Omega} \simeq 0$, which makes the dynamics vastly more interesting.

6.1 The restricted three-body problem

Over the following few sections we'll develop these ideas with application to the restricted three-body problem of a test particle (such as an asteroid or Pluto), moving in the combined potential of the sun plus a large planet (such as Jupiter or Neptune). We have already worked out the full Hamiltonian (4.29) for this system in the general case, but in the restricted problem we focus on the motion of the test particle, imposing the motion of the more massive sun and planet from without. So our first task is to obtain an effective Hamiltonian for the test particle. Let the positions of the sun, large planet and test particle referred to an inertial frame be $\mathbf{x}_0$, $\mathbf{x}_\mathrm{p}$ and $\mathbf{x}$ respectively and their masses be m_0, m_p and m. The equations of motion are

$$m_0\ddot{\mathbf{x}}_0 = \frac{Gm_\mathrm{p}m_0}{|\mathbf{x}_\mathrm{p}-\mathbf{x}_0|^3}(\mathbf{x}_\mathrm{p}-\mathbf{x}_0) + \frac{Gmm_0}{|\mathbf{x}-\mathbf{x}_0|^3}(\mathbf{x}-\mathbf{x}_0),$$

$$m_\mathrm{p}\ddot{\mathbf{x}}_\mathrm{p} = \frac{Gm_0m_\mathrm{p}}{|\mathbf{x}_0-\mathbf{x}_\mathrm{p}|^3}(\mathbf{x}_0-\mathbf{x}_\mathrm{p}) + \frac{Gmm_\mathrm{p}}{|\mathbf{x}-\mathbf{x}_\mathrm{p}|^3}(\mathbf{x}-\mathbf{x}_\mathrm{p}),$$

$$m\ddot{\mathbf{x}} = \frac{Gm_0m}{|\mathbf{x}_0-\mathbf{x}|^3}(\mathbf{x}_0-\mathbf{x}) + \frac{Gm_\mathrm{p}m}{|\mathbf{x}_\mathrm{p}-\mathbf{x}|^3}(\mathbf{x}_\mathrm{p}-\mathbf{x}). \tag{6.6}$$

Now introduce the heliocentric coordinates $\mathbf{r}_\mathrm{p} \equiv \mathbf{x}_\mathrm{p} - \mathbf{x}_0$ and $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}_0$. By taking appropriate combinations of (6.6) we find that

$$\ddot{\mathbf{r}} = -\frac{Gm_0\mathbf{r}}{|\mathbf{r}|^3} + \frac{Gm_\mathrm{p}(\mathbf{r}_\mathrm{p}-\mathbf{r})}{|\mathbf{r}_\mathrm{p}-\mathbf{r}|^3} - \frac{Gm_\mathrm{p}\mathbf{r}_\mathrm{p}}{|\mathbf{r}_\mathrm{p}|^3} \tag{6.7}$$

which corresponds to motion in the Hamiltonian $H(\mathbf{r},\mathbf{p}) = H_0 + \epsilon H_1$, where

$$\begin{aligned} H_0(\mathbf{r},\mathbf{p}) &= \frac{1}{2}\mathbf{p}^2 - \frac{Gm_0}{|\mathbf{r}|}, \\ \epsilon H_1(\mathbf{r};\mathbf{r}_\mathrm{p}) &= -Gm_\mathrm{p}\left(\frac{1}{|\mathbf{r}_\mathrm{p}-\mathbf{r}|} - \frac{\mathbf{r}\cdot\mathbf{r}_\mathrm{p}}{|\mathbf{r}_\mathrm{p}|^3}\right). \end{aligned} \tag{6.8}$$

Here H_0 is just the Kepler Hamiltonian for a central mass m_0. The perturbation H_1 is composed of a direct term $\propto 1/|\mathbf{r}_\mathrm{p}-\mathbf{r}|$ due to the influence of the planet plus an indirect term $\propto \mathbf{r}\cdot\mathbf{r}_\mathrm{p}$ that accounts for the noninertial frame. Notice that H_1 is smaller than H_0 by a factor $\sim m_\mathrm{p}/m_0$, which may take as our definition of the scale ϵ. In the limit $m_\mathrm{p} \to 0$ ($\epsilon \to 0$) the motion reduces to the simple Keplerian case.

[At this point it would make sense to set $Gm_0 = 1$ and $\epsilon = m_\mathrm{p}/m_0$, followed by $a_\mathrm{p} = 1$ below. But let's plod along without doing that, if only to keep our frequencies clear.]

Exercise: Explain how one can obtain (6.8) directly from (4.29). [Hint: remember that (i) $\mathbf{p}_n \neq m_n\dot{\mathbf{r}}_n$ in (4.29); (ii) we are free to add a total time derivative $\mathrm{d}\Lambda/\mathrm{d}t$ to the Hamiltonian, where $\Lambda(\mathbf{q},\mathbf{p})$ is any function of the phase-space coordinates.]

6.2 Example: Lidov–Kozai oscillations

Now consider the even more special situation in which the test particle is much closer to the sun than the planet ($r \ll r_\mathrm{p}$) and suppose that the planet is on a circular orbit, but the test particle

is not confined to the orbital plane of the planet. The following is a simplified version of Tremaine (2014); see also Tremaine & Yavetz (2014).

Use $|\mathbf{r}_\mathrm{p}-\mathbf{r}|^2 = r_\mathrm{p}^2 - 2\mathbf{r}_\mathrm{p}\cdot\mathbf{r} + r^2$ to expand the first term in the perturbation (6.8) as

$$\frac{1}{|\mathbf{r}_\mathrm{p}-\mathbf{r}|} = \frac{1}{r_\mathrm{p}}\left[1 + \frac{2\mathbf{r}_\mathrm{p}\cdot\mathbf{r} - r^2}{2r_\mathrm{p}^2} + \frac{3}{8}\left(\frac{2\mathbf{r}_\mathrm{p}\cdot\mathbf{r} - r^2}{r_\mathrm{p}^2}\right)^2 + \cdots\right]. \tag{6.9}$$

Dropping a constant term, the perturbation is then

$$\epsilon H_1 = -Gm_\mathrm{p}\left(\frac{1}{|\mathbf{r}_\mathrm{p}-\mathbf{r}|} - \frac{\mathbf{r}_\mathrm{p}\cdot\mathbf{r}}{|\mathbf{r}_\mathrm{p}|^3}\right) = \frac{Gm_\mathrm{p}}{r_\mathrm{p}^3}\left[\frac{1}{2}r^2 - \frac{3(\mathbf{r}_\mathrm{p}\cdot\mathbf{r})^2}{2r_\mathrm{p}^2}\right] + O(r^3/r_\mathrm{p}^4). \tag{6.10}$$

We need to average this over the motion of both the test particle and the planet. Let $(\hat{\mathbf{e}}, \hat{\mathbf{u}}, \hat{\mathbf{n}})$ be unit vectors in a right-handed coordinate system centred on the sun with $\hat{\mathbf{e}}$ pointing towards the pericentre of the test particle (which has true anomaly $\varphi = 0$), $\hat{\mathbf{u}}$ pointing towards $\varphi = \pi/2$ and $\hat{\mathbf{n}}$ normal to the test particle's orbit plane. Let $(\hat{\mathbf{e}}_\mathrm{p}, \hat{\mathbf{u}}_\mathrm{p}, \hat{\mathbf{n}}_\mathrm{p})$ be corresponding unit vectors for the planet.[1] Then $\mathbf{r} = r(\varphi)(\cos\varphi\hat{\mathbf{e}} + \sin\varphi\hat{\mathbf{u}})$ and $\mathbf{r}_\mathrm{p} = r_\mathrm{p}(\varphi_\mathrm{p})(\cos\varphi_\mathrm{p}\hat{\mathbf{e}}_\mathrm{p} + \sin\varphi_\mathrm{p}\hat{\mathbf{u}}_\mathrm{p})$. Averaging (6.10) gives

$$\begin{aligned}\epsilon\bar{H}_1 = \frac{Gm_\mathrm{p}}{a_\mathrm{p}^3}\Bigg[&\frac{1}{2}\langle r^2\rangle - \frac{3\langle\cos^2\varphi_\mathrm{p}\rangle}{2a_\mathrm{p}^2}\left(\langle r^2\cos^2\varphi\rangle(\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 + \langle r^2\sin^2\varphi\rangle(\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2\right)\\ &- \frac{3\langle\sin^2\varphi_\mathrm{p}\rangle}{2a_\mathrm{p}^2}\left(\langle r^2\cos^2\varphi\rangle(\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 + \langle r^2\sin^2\varphi\rangle(\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2\right)\Bigg],\end{aligned} \tag{6.11}$$

where the triangular brackets denote time averages. By assumption the planet is on a circular orbit. So, $\langle\cos^2\varphi_\mathrm{p}\rangle = \langle\sin^2\varphi_\mathrm{p}\rangle = \frac{1}{2}$. Similarly, going back to (2.23), we have that

$$\langle r^2\rangle = a^2\left(1 + \frac{3}{2}e^2\right), \quad \langle r^2\cos^2\varphi\rangle = \frac{1}{2}a^2(1+4e^2), \quad \langle r^2\sin^2\varphi\rangle = \frac{1}{2}a^2(1-e^2), \tag{6.12}$$

and so the averaged perturbation becomes

$$\begin{aligned}\epsilon\bar{H}_1 &= \frac{Gm_\mathrm{p}a^2}{4a_\mathrm{p}^3}\Bigg[2 + 3e^2 - \frac{3}{2}(1+4e^2)(\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 - \frac{3}{2}(1-e^2)(\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2\\ &\qquad - \frac{3}{2}(1+4e^2)(\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 - \frac{3}{2}(1-e^2)(\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2\Bigg]\\ &= \frac{Gm_\mathrm{p}a^2}{4a_\mathrm{p}^3}\left[2 + 3e^2 - \frac{3}{2}(1+4e^2)\left((\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 + (\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2\right) - \frac{3}{2}(1-e^2)\left((\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2 + (\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2\right)\right].\end{aligned} \tag{6.13}$$

Now use the relations $(\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 + (\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 + (\mathbf{n}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 = 1$ and $(\hat{\mathbf{e}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2 + (\hat{\mathbf{u}}_\mathrm{p}\cdot\hat{\mathbf{u}})^2 + (\mathbf{n}_\mathrm{p}\cdot\hat{\mathbf{u}})^2 = 1$ to eliminate the dependence of ϵH_1 on $\hat{\mathbf{e}}_\mathrm{p}$ and $\hat{\mathbf{u}}_\mathrm{p}$. The result is

$$\epsilon\bar{H}_1 = \frac{Gm_\mathrm{p}a^2}{4a_\mathrm{p}^3}\left[-1 - \frac{3}{2}e^2 + \frac{3}{2}(1+4e^2)(\mathbf{n}_\mathrm{p}\cdot\hat{\mathbf{e}})^2 + \frac{3}{2}(1-e^2)(\mathbf{n}_\mathrm{p}\cdot\hat{\mathbf{u}})^2\right]. \tag{6.14}$$

We define the orbital elements of the test particle with reference to the orbital plane of the planet. Then $\hat{\mathbf{n}}_\mathrm{p}\cdot\hat{\mathbf{e}} = \sin i\sin\omega$ and $\hat{\mathbf{n}}_\mathrm{p}\cdot\hat{\mathbf{u}} = \sin i\cos\omega$ from (2.20), and

$$\begin{aligned}\epsilon\bar{H}_1 &= \frac{Gm_\mathrm{p}a^2}{8a_\mathrm{p}^3}\left[1 - 6e^2 - 3(1-e^2)\cos^2 i + 15e^2(1-\cos^2 i)\sin^2\omega\right]\\ &= \frac{Gm_\mathrm{p}a^2}{8a_\mathrm{p}^3}\left[-5 + 6\frac{J_2^2}{J_3^2} - 3\frac{J_1^2}{J_3^2} + 15\left(1 - \frac{J_2^2}{J_3^2}\right)\left(1 - \frac{J_1^2}{J_2^2}\right)\sin^2\theta_2\right],\end{aligned} \tag{6.15}$$

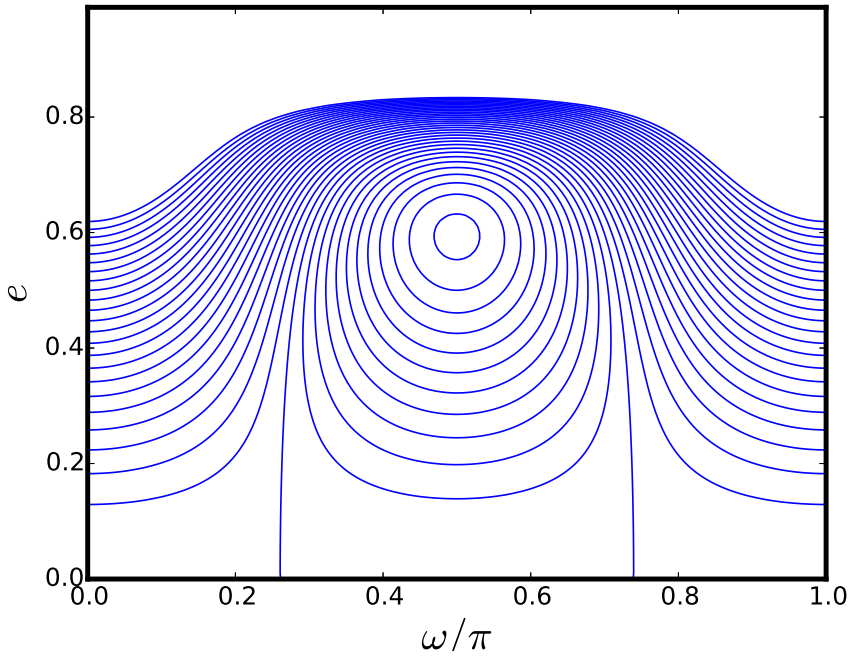

Figure 6.1: **Lidov–Kozai oscillations** Level surfaces of the Lidov–Kozai averaged Hamiltonian (6.17) for $C = \frac{1}{2}$.

when expressed in terms of the Delaunay elements (5.60): recall that J_2 is the total angular momentum, which you'd normally call L (but which celestial mechanicians might label G; they keep the label L for J_3.).

The unperturbed Hamiltonian $H_0 = H_0(J_3)$. Averaging $H = H_0 + \epsilon\bar{H}_1$ over θ_3 we're left with $J_3 = \sqrt{GMa}$ = constant and $J_1 = \sqrt{GMa(1-e^2)}\cos i$ = constant. The dynamics are then determined by the level surfaces of (6.15) as a function of (θ_2, J_2). The semimajor axis a is constant, but orbits can trade eccentricity e for inclination i, with

$$\cos i = \cos i_0 \sqrt{\frac{1-e_0^2}{1-e^2}}, \tag{6.16}$$

where (e_0, i_0) are the initial values of (e, i). Eliminating $\cos i$ in favour of e, the averaged Hamiltonian (6.15) becomes

$$\epsilon\bar{H}_1 = \frac{Gm_{\rm p}a^2}{8a_{\rm p}^3}\left[1 - 6e^2 - 3C + 15e^2\left(1 - \frac{C}{1-e^2}\right)\sin^2\omega\right], \tag{6.17}$$

where the constant $C = (1-e_0^2)\cos^2 i_0$. Figure 6.1 shows the level surfaces of this $\epsilon\bar{H}_1$ for $(e_0, i_0) = (0, 60°)$.

Suppose that the test particle starts on a circular orbit $e_0 = 0$ with inclination i_0. Conservation of $\epsilon\bar{H}_1$ implies either that $e = 0$ (particle remains on a circular orbit) or that

$$2(1-e^2) + 5(e^2 - \sin^2 i_0)\sin^2\omega = 0, \tag{6.18}$$

in which case it executes Lidov–Kozai oscillations about $\omega = \pm 90°$. It is easy to see from (6.18) that the condition for these oscillations to be possible is that $\sin^2 i_0 > 2/5$, or $i_0 > 39.2°$. The maximum eccentricity of $e_{\rm max} = [(5\sin^2 i_0 - 2)/3]^{1/2}$ occurs at $\omega = \pm 90°$.

[1] Made explicit here in case you feel inspired to generalize to the case in which the perturbing planet is on a noncircular orbit...

Notice that the existence of these oscillations is independent of the strength of the perturbation from the distant planet: changing the prefactor in (6.17) just changes the period of the oscillations. So, an arbitrarily small external force can in principle excite these oscillations.

Perhaps the most "down-to-earth" application of these oscillations is the stability of artificial satellites orbiting the earth. There the moon acts as the "distant perturber", but a more complete model of their motion would account for the additional perturbation due to the flattening of the earth: see Tremaine & Yavetz (2014).

6.3 Another example: when average is not good enough

Now let us consider a different regime of the restricted three-body problem. Suppose that the test particle and planet both move in the same plane, but relax the assumption that the test particle is much closer to the central star than the planet. Let $\phi = \cos^{-1}(\mathbf{r}\cdot\mathbf{r}_\mathrm{p}/rr_\mathrm{p})$ be the angle between them. The perturbation from the planet is then, using the cosine rule,

$$\epsilon H_1 = -\frac{Gm_\mathrm{p}}{r_\mathrm{p}}\left[\left(1+\left(\frac{r}{r_\mathrm{p}}\right)^2 - 2\left(\frac{r}{r_\mathrm{p}}\right)\cos\phi\right)^{-1/2} - \left(\frac{r}{r_\mathrm{p}}\right)\cos\phi\right]. \tag{6.19}$$

We want to be able to express this in terms of the angle-action variables for H_0, or, equivalently, in terms of the semimajor axis a, eccentricity e, argument of pericentre and mean anomaly.

We can remove the explicit time dependence by adding a term $\Omega_{\lambda_\mathrm{p}}J_{\lambda_\mathrm{p}}$ to our unperturbed H_0 and extending the phase space to include λ_p. The full Hamiltonian is therefore

$$H(\lambda,\varpi,J_\lambda,J_\varpi,\lambda_\mathrm{p}) = \underbrace{-\frac{1}{2}\left(\frac{Gm_0}{J_\lambda}\right)^2 + \Omega_{\lambda_\mathrm{p}}J_{\lambda_\mathrm{p}}}_{H_0} \underbrace{-\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\sum_{jkl} C_{jkl}(J_\lambda,J_\varpi;a_\mathrm{p},e_\mathrm{p})\mathrm{e}^{\mathrm{i}(j\varpi+k\lambda+l\lambda_\mathrm{p})}}_{\epsilon H_1}, \tag{6.20}$$

in which the coefficients C_{jkl} are found by simply(?) Fourier expanding the perturbation (6.19) in the angles $(\varpi,\lambda,\lambda_\mathrm{p})$. The details of how to do that are left as an exercise at the end of this chapter, but for now let's look at the equations of motion for the Fourier-expanded Hamiltonian (6.20). They are

$$\begin{aligned}
\dot{J}_\varpi &= -\frac{\partial H}{\partial(-\varpi)} = -\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\mathrm{i}\sum_{jkl} jC_{jkl}\mathrm{e}^{\mathrm{i}(j\varpi+k\lambda+l\lambda_\mathrm{p})},\\
\dot{J}_\lambda &= -\frac{\partial H}{\partial\lambda} = +\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\mathrm{i}\sum_{jkl} kC_{jkl}\mathrm{e}^{\mathrm{i}(j\varpi+k\lambda+l\lambda_\mathrm{p})},\\
-\dot{\varpi} = \dot{\theta}_\varpi &= \frac{\partial H}{\partial J_\varpi} = -\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\sum_{jkl}\frac{\partial C_{jkl}}{\partial J_\varpi}\mathrm{e}^{\mathrm{i}(j\varpi+k\lambda+l\lambda_\mathrm{p})},\\
\dot{\lambda} = \dot{\theta}_\lambda &= \frac{\partial H}{\partial J_\lambda} = \frac{[Gm_0]^2}{J_\lambda^3} - \frac{Gm_\mathrm{p}}{a_\mathrm{p}}\sum_{jkl}\frac{\partial C_{jkl}}{\partial J_\lambda}\mathrm{e}^{\mathrm{i}(j\varpi+k\lambda+l\lambda_\mathrm{p})},
\end{aligned} \tag{6.21}$$

If we integrate, say, the first of these equations in the usual way by substituting the unperturbed motion $\lambda = \Omega_\lambda t + \text{const}$, $\lambda_\mathrm{p} = \Omega_{\lambda_\mathrm{p}}t + \text{const}$ into the RHS, we obtain

$$J_\varpi(t) = -\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\sum_{jkl}\frac{jC_{jkl}}{k\Omega_\lambda + l\Omega_{\lambda_\mathrm{p}}}\mathrm{e}^{\mathrm{i}(k\Omega_\lambda+l\Omega_{\lambda_\mathrm{p}})t}\mathrm{e}^{\mathrm{i}(j\varpi+\text{const})} + \text{const}. \tag{6.22}$$

The denominator in this expression means that the supposedly small correction to J_ϖ blows up whenever the frequencies Ω_λ and $\Omega_{\lambda_\mathrm{p}}$ are close to resonance. The reason for this misbehaviour is simple: the (j,k,l) component of the perturbation acts for a period $T = 2\pi/(j\Omega_\varpi + k\Omega_\lambda + l\Omega_{\lambda_\mathrm{p}})$; the closer we are to resonance, the longer this perturbation is applied and the stronger the deviation of the resulting orbit from its first-order unperturbed form.

6.4 How to treat orbits close to resonance?

Consider the case in which Ω and Ω_p are related through $K\Omega \simeq K_\mathrm{p}\Omega_\mathrm{p}$, where K and K_P are integers. That is, for every K orbits the planet makes, the test body completes K_p. This is known as a $K_\mathrm{p} : K$ mean-motion resonance. For example, the Pluto–Neptune system has $3\Omega = 2\Omega_\mathrm{p}$, so that $K = 3$, $K_\mathrm{p} = 2$: they are in a $2 : 3$ mean-motion resonance. Another example is provided by the Hilda asteroids, which are in a $3 : 2$ resonance with Jupiter. We assume that $K \neq K_\mathrm{p}$ in the following.

In the expansion (6.20) of $H_0 + \epsilon H_1$ let us focus on the $(j, k, l) = (0, 0, 0)$ term plus the $(j, k, l) = (j, K, -K_\mathrm{p})$ and $(j, -K, K_\mathrm{p})$ resonant terms that lead to the small denominators problem. Some good news: we need only a very restricted range of values of j. To see this, consider rotating our coordinate system by an angle $\Delta\varphi$ about the z axis. This increases the Euler angle Ω by $\Delta\varphi$, which, from (5.64), means that ϖ, λ and λ_p increase by the same amount. But ϵH_1 in (6.20) must be unchanged under such rotations. So we must have $j+k+l = 0$. That is, $j = -(k+l) = -(K-K_\mathrm{p})$ when $k = +K$, $l = -K_\mathrm{p}$, and $j = (K - K_\mathrm{p})$ when $k = -K$.[2] Then the troublesome, resonant part of the perturbation can be written as

$$\epsilon H_{1,\mathrm{res}} = -\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\left[A(\mathbf{J}) + B(\mathbf{J})\cos((K_\mathrm{p} - K)\varpi + K\lambda - K_\mathrm{p}\lambda_\mathrm{p} + \beta)\right], \tag{6.23}$$

where $A = C_{000}$ and the amplitude B and phase β of the cosine term are set by the pair of functions $C_{jkl}(\mathbf{J})$ having $(j, k, l) = \pm(K_\mathrm{p} - K, K, -K_\mathrm{p})$.

Now we make a couple of transformations to new angle–action variables. The resonance occurs when J_λ is equal to

$$J_\mathrm{res} \equiv \left(\frac{K}{K_\mathrm{p}}\frac{(Gm_0)^2}{\Omega_{\lambda_\mathrm{p}}}\right)^{1/3}. \tag{6.24}$$

Introduce

$$\Delta J_\lambda = J_\lambda - J_\mathrm{res} \tag{6.25}$$

to measure how far we are from the resonance, and let $\Delta\lambda = \lambda + \frac{1}{K}\beta$, so that the resonant part of the perturbation becomes

$$\epsilon H_{1,\mathrm{res}} = -\frac{Gm_\mathrm{p}}{a_\mathrm{p}}\left[A(\mathbf{J}) + B(\mathbf{J})\cos\left((K_\mathrm{p} - K)\varpi + K\Delta\lambda - K_\mathrm{p}\lambda_\mathrm{p}\right)\right]. \tag{6.26}$$

Then transform to new angle–action coordinates $(\theta_\mathrm{s}, \theta_\mathrm{f1}, \theta_\mathrm{f2}, J_\mathrm{s}, J_\mathrm{f1}, J_\mathrm{f2})$ in which θ_s is defined by the argument of the cosine. For example, for the $K_\mathrm{p} : K = 2 : 3$ resonance of the Neptune–Pluto system, we can take

$$\begin{pmatrix}\theta_\mathrm{s}\\ \theta_\mathrm{f1}\\ \theta_\mathrm{f2}\end{pmatrix} = \begin{pmatrix}1 & 3 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\varpi\\ \Delta\lambda\\ \lambda_\mathrm{p}\end{pmatrix}, \quad \begin{pmatrix}J_\varpi\\ \Delta J_\lambda\\ J_{\lambda_\mathrm{p}}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 0 & 1\end{pmatrix}\begin{pmatrix}J_\mathrm{s}\\ J_\mathrm{f1}\\ J_\mathrm{f2}\end{pmatrix}, \tag{6.27}$$

so that the resonant perturbation becomes simply

$$\epsilon H_1 = -\frac{Gm_\mathrm{p}}{a_\mathrm{p}}[A(\mathbf{J}) + B(\mathbf{J})\cos\theta_\mathrm{s}], \tag{6.28}$$

which depends only on the "slow" angle $\theta_\mathrm{s} = (3\Delta\lambda - 2\lambda_\mathrm{p}) - \varpi$. Adding H_0, the full resonant Hamiltonian (6.20) in these new coordinates is given by

$$H_\mathrm{res} = -\frac{(Gm_0)^2}{2\underbrace{(3J_\mathrm{s} + J_\mathrm{f1} + J_\mathrm{res})^2}_{=(\Delta J_\lambda + J_\mathrm{res})^2}} + \Omega_{\lambda_\mathrm{p}}\underbrace{(J_\mathrm{f2} - 2J_\mathrm{s})}_{=J_{\lambda_\mathrm{p}}} - \frac{Gm_\mathrm{p}}{a_\mathrm{p}}[A(\mathbf{J}) + B(\mathbf{J})\cos\theta_\mathrm{s}], \tag{6.29}$$

plus a remainder term $H_\mathrm{remainder}$ that includes all the other (j, k, l) terms from our original ϵH_1. Ignoring this $H_\mathrm{remainder}$ we see that the "fast" variables J_f1, J_f2 are integrals of motion in H_res. So

[2] This is one of the so-called d'Alembert rules. See §1.9.3 of Morbidelli (2002).

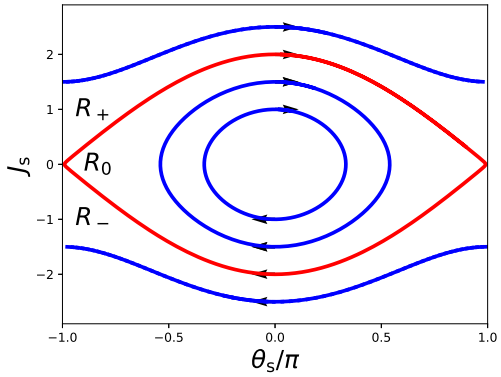

Figure 6.2: Phase plane for the pendulum Hamiltonian (6.30). There are fixed points at $(\theta_s, J_s) = (0,0)$ (stable) and $(\pi, 0)$ (unstable). The separatrix orbit (red) divides the plane into three regions, which we label R_-, R_0 and R_+ according to the sign of $\dot{\theta}_s$. Inside the separatrix (R_0) orbits *librate* around the stable fixed point, with $\dot{\theta}_s$ having boths signs. Outside the separatrix in regions R_+ and R_- they *circulate* with $\dot{\theta}_s > 0$ and $\dot{\theta}_s < 0$, respectively.

the behaviour of the system is completely determined by the level surfaces of $H_{\rm res}(\theta_s, J_s)$, with J_{f1} and J_{f2} playing the role of parameters.

The part of $H_{\rm res}$ that is independent of θ_s will have an extremum for some $J_s = J_{s,0}$. Writing $\Delta J_s \equiv J_s - J_{s,0}$ and Taylor expanding, then, provided $B(J_{s,0}) \neq 0$, we can approximate

$$H_{\rm res}(\theta_s, \Delta J_s) = \text{const} + C\Delta J_s^2 - \frac{Gm_1}{a_p} B\cos\theta_s + O(\Delta J_s^3), \tag{6.30}$$

where B and C are independent of θ_s and ΔJ_s. This is the Hamiltonian of a pendulum. Its phase plane is shown in Figure 6.2. Suppose that $BC > 0$. Then $H_{\rm res}$ has two equilibrium points: a stable one (an *elliptic* fixed point) at $(\theta_s, \Delta J_s) = (0,0)$ and an unstable one (*hyperbolic* fixed point) at $(\theta_s, \Delta J_s) = (\pm\pi, 0)$. The red curve in Figure 6.2 that joins the unstable fixed point to its periodic image is the **separatrix**; it separates the $(\theta_s, \Delta J_s)$ phase plane into three sections. Above the separatrix (region R_+) the angle θ_s **circulates** with $\dot{\theta}_s > 0$. Within the separatrix (region R_0) θ_s **librates** about the stable equilibrium point $\theta_s = 0$. Below the separatrix (region R_-) θ_s circulates, but with $\dot{\theta}_s < 0$. Pluto avoids colliding with Neptune by librating around the stable equilibrium position of θ_s.

6.5 Perturbation theory by canonical maps

Let's consider a slightly more sophisticated way of treating the perturbation Hamiltonian ϵH_1. In general, the perturbed problem can be described by a Hamiltonian

$$H^0(\boldsymbol{\theta}^0, \mathbf{J}^0) = H_0(\mathbf{J}^0) + \epsilon H_1(\boldsymbol{\theta}^0, \mathbf{J}^0), \tag{6.31}$$

where we are able to construct angle-action coordinates $(\boldsymbol{\theta}^0, \mathbf{J}^0)$ for some unperturbed Hamiltonian H_0, and ϵ is a small parameter that controls the size of the perturbation H_1. For solar system problems H_0 is the Kepler Hamiltonian due to the Sun. The dominant contribution in H_1 is the perturbation due to Jupiter, which has a mass $10^{-3}\, M_\odot$. Therefore $\epsilon \sim 10^{-3}$.

Idea: look for a canonical map, close to the identity, of the form

$$\boldsymbol{\theta}^0 = \boldsymbol{\theta}^1 + \epsilon\Delta\boldsymbol{\theta}^1(\boldsymbol{\theta}^1, \mathbf{J}^1), \quad \mathbf{J}^0 = \mathbf{J}^1 + \epsilon\Delta\mathbf{J}^1(\boldsymbol{\theta}^1, \mathbf{J}^1), \tag{6.32}$$

in terms of which the Hamiltonian, expressed as a function of the new coordinates,

$$H^1(\boldsymbol{\theta}^1, \mathbf{J}^1) = H_0(\mathbf{J}^1 + \epsilon\Delta\mathbf{J}^1) + \epsilon H_1(\boldsymbol{\theta}^1 + \epsilon\Delta\boldsymbol{\theta}^1, \mathbf{J}^1 + \epsilon\Delta\mathbf{J}^1), \tag{6.33}$$

is independent of $\boldsymbol{\theta}^1$ to first order in ϵ:

$$H^1(\boldsymbol{\theta}^1, \mathbf{J}^1) = H_0(\mathbf{J}^1) + \epsilon \bar{H}_1(\mathbf{J}^1) + \epsilon^2 H_2(\boldsymbol{\theta}^1, \mathbf{J}^1). \tag{6.34}$$

For instance, $\bar{H}_1$ could be obtained by averaging (6.33) over $\boldsymbol{\theta}^1$, with H_2 scooping up the leftovers. Then, the equations of motion for our new first-order Hamiltonian $H_0(\mathbf{J}^1) + \epsilon \bar{H}_1(\mathbf{J}^1)$ are simply $\dot{\mathbf{J}}^1 = 0$ and $\dot{\boldsymbol{\theta}}^1 = \frac{\partial}{\partial \mathbf{J}^1}(H_0 + \epsilon \bar{H}_1)$, which can be transformed back to our original $(\boldsymbol{\theta}^0, \mathbf{J}^0)$ coordinates using (6.32)

We can iterate this procedure. Let

$$\boldsymbol{\theta}^{r-1} = \boldsymbol{\theta}^r + \epsilon \Delta\boldsymbol{\theta}^r(\boldsymbol{\theta}^r, \mathbf{J}^r), \quad \mathbf{J}^{r-1} = \mathbf{J}^r + \epsilon \Delta\mathbf{J}^r(\boldsymbol{\theta}^r, \mathbf{J}^r). \tag{6.35}$$

Then

$$H^r(\boldsymbol{\theta}^r, \mathbf{J}^r) = H_0(\mathbf{J}^r) + \epsilon \bar{H}_1(\mathbf{J}^r) + \cdots + \epsilon^r \bar{H}_r(\mathbf{J}^r) + \epsilon^{r+1} H_{r+1}(\boldsymbol{\theta}^r, \mathbf{J}^r), \tag{6.36}$$

with each $\bar{H}_r$ and H_{r+1} obtained by averaging the preceding $H^{r-1}(\boldsymbol{\theta}^{r-1}, \mathbf{J}^{r-1})$ over the new $\boldsymbol{\theta}^r$.

Unfortunately this simple procedure does not work up to arbitrarily high order: resonances are unavoidable. A more immediate problem is that we need to ensure that the maps (6.32) and (6.35) are canonical. Let's consider these in turn.

6.5.1 Lie transforms

In doing this we need to make sure that the maps (6.32) and (6.35) are canonical. The neat way of achieving this is to use the fact that in the limit $\epsilon \to 0$ they reduce to the identity map. That is, they must be generated by some phase flow. Write $\mathbf{w}^r = (\boldsymbol{\theta}^r, \mathbf{J}^r)$. Then we can express (6.35) as

$$\mathbf{w}^{r-1} = \exp[\epsilon \mathcal{L}_{\chi_r}] \mathbf{w}^r \tag{6.37}$$

and the problem reduces to finding a generating function χ_r that takes H^r to H^{r-1}. We focus on the first such generating function, χ_1, and write $\mathbf{w}^0(\epsilon) = \mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} \mathbf{w}^1$.

> **Exercise:** The *Lie transform* of the function $f(\mathbf{w})$ under the flow generated by another function χ is defined as $\tilde{f}(\mathbf{w}) \equiv f(\mathrm{e}^{\epsilon \mathcal{L}_\chi} \mathbf{w})$. Show that $\tilde{f} = \exp[\epsilon \mathcal{L}_\chi] f$.

Now use the fact that H^0 and H^1 refer to the the same perturbed Hamiltonian, just expressed in different coordinates. To make this distinction clear, write H (unadorned, without superscript) for the underlying (coordinate-free) Hamiltonian function and introduce a pair of (symplectic) coordinate maps, φ_0 and φ_1, so that

$$\begin{aligned} H^0(\mathbf{w}^0) &= H \circ \varphi_0(\mathbf{w}^0), \\ H^1(\mathbf{w}^1) &= H \circ \varphi_1(\mathbf{w}^1). \end{aligned} \tag{6.38}$$

Clearly $\varphi_0(\mathbf{w}^0) = \varphi_1(\mathbf{w}^1)$. Now consider two different ways of looking at the flow generated by χ_1. One is that χ_1 has no effect on H itself, but instead changes the phase-space location of the point $P = \varphi_0(\mathbf{w}^0) = \varphi_1(\mathbf{w}^1)$ that is fed to H. The output of the function after the flow has acted is then $H \circ \varphi_1(\mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} \mathbf{w}^1)$. Another view is that the flow leaves P unchanged but modifies H, changing it to a new function $\mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} H$. In that case the result of applying the function is $(\mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} H) \circ \varphi_1(\mathbf{w}^1)$. These two outputs should be equal. So,

$$H \circ \varphi_1(\mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} \mathbf{w}^1) = (\mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} H) \circ \varphi_1(\mathbf{w}^1). \tag{6.39}$$

Using $\mathbf{w}^0 = \mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} \mathbf{w}^1$ and $\varphi_1(\mathbf{w}^1) = \varphi_0(\mathbf{w}^0)$ this becomes

$$H \circ \varphi_1(\mathbf{w}^0) = (e^{\epsilon \mathcal{L}_{\chi_1}} H) \circ \varphi_0(\mathbf{w}^0). \tag{6.40}$$

Replacing the argument $\mathbf{w}^0$ by $\mathbf{w}^1 = (\boldsymbol{\theta}^1, \mathbf{J}^1)$ and switching back to more conventional notation, we obtain our formal expression for H expressed in terms of the new coordinates:

$$H^1(\boldsymbol{\theta}^1, \mathbf{J}^1) = \mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} H^0(\boldsymbol{\theta}^1, \mathbf{J}^1). \tag{6.41}$$

Substitute $H^0 = H_0 + \epsilon H_1$ and expand, giving

$$\begin{aligned} H^1 &= H^0 + \epsilon[H^0, \chi_1] + \frac{1}{2}\epsilon^2[[H^0, \chi_1], \chi_1] + O(\epsilon^3) \\ &= H_0 + \epsilon H_1 + \epsilon[H_0, \chi_1] + \epsilon^2[H_1, \chi_1] + \frac{1}{2}\epsilon^2[[H_0, \chi_1], \chi_1] + O(\epsilon^3), \end{aligned} \tag{6.42}$$

in which everything has arguments $(\boldsymbol{\theta}^1, \mathbf{J}^1)$.

6.5.2 Averaging

Look at the $O(\epsilon)$ terms in this expression. Comparing to (6.34), the $O(\epsilon)$ factors agree if we take

$$H_1 + [H_0, \chi_1] = \bar{H}_1(\mathbf{J}^1). \tag{6.43}$$

To find this generating Hamiltonian χ_1, expand H_1 and χ_1 as Fourier series in the (new) angles:

$$\begin{aligned} H_1(\boldsymbol{\theta}^1, \mathbf{J}^1) &= \sum_{\mathbf{n}} c_{\mathbf{n}}(\mathbf{J}^1) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}^1}, \\ \chi_1(\boldsymbol{\theta}^1, \mathbf{J}^1) &= \sum_{\mathbf{n}} d_{\mathbf{n}}(\mathbf{J}^1) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}^1}. \end{aligned} \tag{6.44}$$

Using $H_0 = H_0(\mathbf{J}^1)$ from (6.34) and defining $\boldsymbol{\Omega}_0(\mathbf{J}^1) \equiv \partial H_0/\partial \mathbf{J}^1$, we find that

$$[H_0, \chi_1] = -\mathrm{i} \sum_{\mathbf{n}} d_{\mathbf{n}}(\mathbf{J}^1) \mathbf{n} \cdot \boldsymbol{\Omega}_0(\mathbf{J}^1) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}^1}. \tag{6.45}$$

Then choosing coefficients

$$d_{\mathbf{0}} = 0, \quad d_{\mathbf{n}}(\mathbf{J}^1) = -\mathrm{i} \frac{c_{\mathbf{n}}(\mathbf{J}^1)}{\mathbf{n} \cdot \boldsymbol{\Omega}_0(\mathbf{J}^1)} \tag{6.46}$$

gives $H_1 + [H_0, \chi_1] = c_0$. That is, the mapping generated by the function χ_1 in (6.44) with coefficients (6.46) annihilates the first-order dependence of H_1 on the angles. The function

$$\bar{H}_1(\mathbf{J}^1) = c_{\mathbf{0}}(\mathbf{J}^1), \tag{6.47}$$

which is just the perturbation $H_1(\boldsymbol{\theta}^1, \mathbf{J}^1)$ averaged over $\boldsymbol{\theta}^1$.

6.5.3 Higher order

Notice the $\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}^1)$ in the denominator in the expression for $d_{\mathbf{n}}$: resonances or almost resonances between the components of the frequency vector $\boldsymbol{\Omega}_0$ leads to the notorious "small denominators" problem. Let's ignore that for now and continue to second order by applying $\exp[\epsilon^2 \mathcal{L}_{\chi_2}]$ to the transformed first-order Hamiltonian (6.42). The result is

$$H^2 = H_0 + \epsilon \bar{H}_1 + \epsilon^2 \left(H_2 + [H_0, \chi_2]\right) + \epsilon^3 [\bar{H}_1, \chi_2] + O(\epsilon^4), \tag{6.48}$$

and we can eliminate the $\boldsymbol{\theta}$ dependence of the $O(\epsilon^2)$ term by choosing χ_2 to satisfy $H_2 + [H_0, \chi_2] = \bar{H}_2(\mathbf{J}^2)$, the angle-averaged H_2. Similarly, we could proceed order by order following (6.35), choosing χ_3 to eliminate the angle-dependence of the $O(\epsilon^3)$ perturbation, and so on.

We can do better than this, however. Notice how simple the $O(\epsilon^3)$ term in (6.48) is. That means we can choose χ_3 at the same time as χ_2, pushing the angle dependence of the transformed Hamiltonian to its $O(\epsilon^4)$ term. Then we can choose $\chi_4, ..., \chi_7$ to make the leading angle-dependent term be $O(\epsilon^8)$, and so on (see §§2.5,2.6 of Lichtenberg & Lieberman (1992) for details or the 2017 miniproject for an example). This is one of two key ingredients used in showing that convergent perturbation series expansions can be found sufficiently far from resonances. The other is explained in Appendix 8 of Arnold (1978); see also Box 3.5 of Binney & Tremaine (2008) or §2.3 of Morbidelli (2002)

6.6 Adiabatic invariants

Now we turn to the special case of how Hamiltonian systems respond to perturbations that develop on timescales much slower than any of the characteristic frequencies $\boldsymbol{\Omega}$. Let us begin with a system having one degree of freedom and corresponding angle-action coordinates (θ, J). Applying a slow perturbation $\epsilon H_1(\theta, J, \epsilon t)$, the Hamiltonian of the system becomes

$$H(\theta, J, t) = H_0(J) + \epsilon H_1(\theta, J, \epsilon t). \tag{6.49}$$

We assume that the derivatives $\partial H_1/\partial\theta \sim \partial H_1/\partial J \sim |H_1| \sim |H_0|$; the ϵ factor in the time dependence of H_1 is chosen to ensure that $\partial H_1/\partial t \sim \epsilon|H_1|$ too.

Now we construct a time-dependent canonical map to new angle-action coordinates (θ^1, J^1) in which to $O(\epsilon)$ the transformed Hamiltonian is independent of θ^1. We use the type-2 generating function

$$F_2(\theta, J^1, t) = \theta J^1 + \epsilon S(\theta, J^1, t), \tag{6.50}$$

which, to first order in ϵ, maps

$$\theta = \theta^1 - \epsilon\frac{\partial S}{\partial J^1}, \quad J = J^1 + \epsilon\frac{\partial S}{\partial \theta}, \tag{6.51}$$

and, from equation (5.10), transforms the Hamiltonian to

$$H^1(\theta^1, J^1, t) = H(\theta, J, t) + \epsilon\frac{\partial S(\theta, J^1, t)}{\partial t}. \tag{6.52}$$

Putting these last four equations together, we have that

$$H^1(\theta^1, J^1, t) = H_0(J_1) + \epsilon\frac{\partial H_0(J^1)}{\partial J}\frac{\partial S}{\partial \theta^1} + \epsilon H_1(\theta^1, J^1, \epsilon t) + \epsilon\frac{\partial S(\theta^1, J^1, t)}{\partial t} + O(\epsilon^2). \tag{6.53}$$

Introducing the angle-averaged

$$\bar{H}_1(J^1, \epsilon t) \equiv \frac{1}{2\pi}\int H_1(\theta^1, J^1, \epsilon t)\,\mathrm{d}\theta^1 \tag{6.54}$$

we can eliminate the θ^1 dependence in the $O(\epsilon)$ terms of the transformed Hamiltonian (6.53) by choosing S to make

$$\Omega_0\frac{\partial S}{\partial \theta^1} + (H_1 - \bar{H}_1) = 0, \tag{6.55}$$

where $\Omega_0 \equiv \partial H_0/\partial J$; this choice of S demotes the $\epsilon\partial S/\partial t$ term in (6.53) to second order in ϵ. The new action is given by

$$\begin{aligned} J^1(\theta, J) &= J - \epsilon\frac{\partial S}{\partial \theta} + O(\epsilon^2) \\ &= J + \epsilon\frac{1}{\Omega(J)}(H_1 - \bar{H}_1)(\theta, J) + O(\epsilon^2). \end{aligned} \tag{6.56}$$

That is, to first order in ϵ, the new action just oscillates around its old value. We can make these oscillations arbitrarily small by slowing down the rate of change ϵ of the perturbation.

An **adiabatic invariant** is any function $I(q, p, t)$ of the phase space coordinates and time that satisfies the condition that, for any $\delta > 0$, we can find some $\epsilon_0 > 0$ for which

$$|I(q(t), p(t), \epsilon t) - I(q(0), p(q), 0)| < \delta \tag{6.57}$$

for all $0 < t < 1/\epsilon$ and $\epsilon < \epsilon_0$. The action is an example of an adiabatic invariant, as is any function of the action.

The situation is less clear cut in systems with more than one degree of freedom. Then equation (6.55) becomes

$$\boldsymbol{\Omega}\cdot\frac{\partial S}{\partial \boldsymbol{\theta}^1} + (H_1 - \bar{H}_1) = 0, \tag{6.58}$$

which is subject to the small denominators problem for almost-resonant orbits: when linear combinations of frequencies are low, the perturbation is no longer slow. This can have interesting consequences.

6.7 Capture into resonance

Fourier expanding $S(\boldsymbol{\theta}, \mathbf{J}^1, t) = \sum_{\mathbf{n}} s_{\mathbf{n}}(\mathbf{J}^1, t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ and $(H_1 - \bar{H}_1)(\boldsymbol{\theta}, \mathbf{J}^1, t) = \sum_{\mathbf{n}} h_{\mathbf{n}}(\mathbf{J}^1, t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ in the Hamiltonian (6.53), we encounter the small-denominators problem for any $\mathbf{n}$ for which $\mathbf{n}\cdot\Omega s_{\mathbf{n}} + h_{\mathbf{n}}$ cannot be eliminated. Following §6.4, let us suppose that there is a single dominant such value of $\mathbf{n}$ for some $\mathbf{J}^1$, and introduce the slow angle $\theta_\mathrm{s} = \mathbf{n}\cdot(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ with phase offset $\boldsymbol{\theta}_0 = \arg h_{\mathbf{n}} = -\arg h_{-\mathbf{n}}$. Then the best we can do is reduce (6.53) to the time-dependent pendulum Hamiltonian

$$H = H_0(J_\mathrm{s}) + \epsilon\bar{H}_1(J_\mathrm{s}, \epsilon t) - \epsilon V(J_\mathrm{s}, \epsilon t)\cos\theta_\mathrm{s}, \tag{6.59}$$

where $V = -2|h_{\mathbf{n}}|$, J_s is a slow action conjugate to θ_s and we have suppressed all dependence on the remaining, fast variables. We have already seen the phase-plane for this Hamiltonian (Figure 6.2), but now the location and size of the separatrix change slowly with time. As usual, the closer we get to the separatrix, the lower the orbital frequency $\Omega_\mathrm{s} = \partial H/\partial J_\mathrm{s}$ becomes. So what happens when a resonance sweeps slowly through action space? This was worked out by Henrard (1982). The following summary borrows from §6.3 of Tremaine (2023).

We are interested in the behaviour of orbits close to the separatrix. We do this by defining the relative energy

$$K(\theta_\mathrm{s}, J_\mathrm{s}, \epsilon t) = H(\theta_\mathrm{s}, J_\mathrm{s}, \epsilon t) - H(\theta, J_\mathrm{s}^\pm, \epsilon t),$$

where $J_\mathrm{s}^\pm(\theta_\mathrm{s}, J_\mathrm{s}, \epsilon t)$ is the action of the closest branch of the separatrix to $(\theta_\mathrm{s}, J_\mathrm{s})$: we use $J_\mathrm{s}^+(\theta_\mathrm{s}, \epsilon t)$ for the action of the $\dot\theta_\mathrm{s} > 0$ segment of the separatrix (in the example shown in Figure 6.2 this is the upper segment) and J_s^- for the $\dot\theta_\mathrm{s} < 0$ segment (lower segment in the figure).

Holding $(\theta_\mathrm{s}, J_\mathrm{s})$ fixed, the rate of change of the relative energy is

$$\frac{\partial}{\partial t}K(\theta_\mathrm{s}, J_\mathrm{s}, \epsilon t) = \frac{\partial}{\partial t}H(\theta_\mathrm{s}, J_\mathrm{s}, \epsilon t) - \frac{\partial}{\partial t}H(\theta_\mathrm{s}, J_\mathrm{s}^\pm, \epsilon t) - \frac{\partial J_\mathrm{s}^\pm}{\partial t}\frac{\partial}{\partial J_\mathrm{s}}H(\theta_\mathrm{s}, J_\mathrm{s}^\pm, \epsilon t). \tag{6.60}$$

When J_s is very close to $J_\mathrm{s}^\pm$, Henrard (1982) shows that first two terms almost cancel, leaving only the third. Dividing by $\dot\theta_\mathrm{s} = \partial H/\partial J_\mathrm{s}$, we are left with

$$\frac{\mathrm{d}K}{\mathrm{d}\theta_\mathrm{s}} = -\frac{\partial}{\partial t}J_\mathrm{s}^\pm(\theta_\mathrm{s}, \epsilon t) \tag{6.61}$$

along orbits near the separatrix. Integrating from $\theta_\mathrm{s} = -\pi$ to π, the change in the relative energy K as an orbit crosses either segment of the separatrix is given by

$$\Delta K_\pm = \mp\int_{-\pi}^{\pi}\frac{\partial J_\mathrm{s}^\pm(\theta_\mathrm{s}, \epsilon t)}{\partial t}\,\mathrm{d}\theta_\mathrm{s}, \tag{6.62}$$

in which for orbits just inside the separatrix the integral really goes from just over $-\pi$ to just under π, a detail that Henrard (1982) shows we can safely ignore. More importantly, he addresses the issue of particles lingering for longer and longer as they approach the unstable fixed point at $\theta_\mathrm{s} = \pm\pi$ and shows that this expression nevertheless is a good approximation to the energy change in crossing the separatrix.

For the example plotted in Figure 6.2 the fixed point $(\theta_\mathrm{s}, J_\mathrm{s}) = (0, 0)$ is a minimum of the Hamiltonian. The relative energy K increases from $K < 0$ in region R_0 of trapped, librating orbits, passing though $K = 0$ on the separatrix, to $K > 0$ in the circulating regions R_+ and R_-. ΔK^+ is directly proportional to the orbit-averaged rate at which the upper segment of the separatrix moves *downwards* in the plot. ΔK^- measures the corresponding rate at which the lower segment moves *upwards*. Particles that librate with $\theta_\mathrm{s} > 0$ (region R_+) gain energy ΔK^+ every libration period. Those that librate with $\theta_\mathrm{s} < 0$ (region R_-) gain energy ΔK^-. Particles that circulate (region R_0) close to the separatrix alternately gain ΔK^+ when $\dot\theta_\mathrm{s} > 0$ and then ΔK^- when $\dot\theta_\mathrm{s} < 0$.

Now consider the fate of a particle in region R_+, with $K > 0$ and $\dot\theta_\mathrm{s} > 0$. It will encounter the separatrix only if $\Delta K^+ < 0$. Once it does then K becomes negative, indicating that the particle is in region R_0. When $\dot\theta_\mathrm{s}$ flips from positive to negative then the particle is closer to the lower half of the separatrix and its behaviour is governed by ΔK^-. There are three cases to consider:

1. If $\Delta K^- < 0$ then the lower separatrix is moving away, downwards, and K continues to decrease: the particle becomes trapped in the librating region R_0.

2. If $\Delta K^- > 0$ (lower separatrix moving up) with $\Delta K^+ + \Delta K^- > 0$ then K becomes positive again and the particle tunnels directly from region R_+ to region R_-.

3. If $\Delta K^- > 0$, but $\Delta K^+ + \Delta K^- < 0$ then the particle's fate depends on its relative energy $K^\star$ just as it enters the slow zone where $\theta_s \simeq \pi$. Assume that $K^\star$ is uniformly distributed in the range $[\Delta K^+, 0]$. If $K^\star + \Delta K^- > 0$ then after the particle has moved from $\theta_s = \pi$ back to $\theta_s = -\pi$ it will have $K > 0$ and so will be in region R_-. The probability of this escape is $\Delta K^-/(-\Delta K^+)$. On the other hand, if $K^\star + \Delta K^- < 0$ the particle will have $K < 0$ and will be trapped in region R_0. The probability of this capture is $(\Delta K^- + \Delta K^+)/\Delta K^+$.

We can apply similar reasoning to the fate of a particle that starts in the trapped region R_0. The particle remains trapped there unless $\Delta K^+ + \Delta K^- > 0$ (i.e., unless the trapped region is shrinking). Then

1. if $\Delta K^+ > 0$ and $\Delta K^- < 0$ (both segments of separatrix moving downwards) it escapes to region R_+;

2. if $\Delta K^+ < 0$ and $\Delta K^- > 0$ (both segments moing upwards) it escapes to region R_-;

3. if $\Delta K^+ > 0$ and $\Delta K^- > 0$ it escapes to region R_+ with probability $\Delta K^+/(\Delta K^+ + \Delta K^-)$ and to region R_- with probability $\Delta K^-/(\Delta K^+ + \Delta K^-)$.

This resonant trapping mechanism is the most plausible explanation for the origin of the 3:2 Neptune–Pluto resonance: see §6.4 of Tremaine (2023) for discussion. It has applications in galaxy dynamics too. Sridhar & Touma (1996) have shown how it can lead to vertical thickening of galactic discs, although in practice scattering of stars by molecular clouds is a much more effective fattening mechanism. Chiba et al. (2021) and Chiba & Schönrich (2021) discuss the observational signatures of orbit trapping by the slowing Galactic bar.

Exercises

1. Let $(\boldsymbol{\theta}^0, \mathbf{J}^0)$ be angle-action variables for the Hamiltonian $H_0(\mathbf{J}^0)$. We wish to construct approximate angle-action variables $(\boldsymbol{\theta}^1, \mathbf{J}^1)$ for the perturbed Hamiltonian $H = H_0 + \epsilon H_1$, where $\epsilon H_1(\boldsymbol{\theta}^0, \mathbf{J}^0)$ is a known function of the $(\boldsymbol{\theta}^0, \mathbf{J}^0)$. Explain why it is appropriate to construct these new $(\boldsymbol{\theta}^1, \mathbf{J}^1)$ by using a type-2 generating function of the form $F_2(\boldsymbol{\theta}^1, \mathbf{J}^0) = \boldsymbol{\theta}^1 \cdot \mathbf{J}^0 + \epsilon S(\boldsymbol{\theta}^1, \mathbf{J}^0)$. What is the relationship between $(\boldsymbol{\theta}^0, \mathbf{J}^0)$ and $(\boldsymbol{\theta}^1, \mathbf{J}^1)$ in terms of the function S? Obtain an expression for $H = H_0 + \epsilon H_1$ as a function of $\boldsymbol{\theta}^1, \mathbf{J}^1$, correct to first order in ϵ. By Fourier expanding

$$S(\boldsymbol{\theta}^1, \mathbf{J}^0) = \sum_{\mathbf{n}} S_{\mathbf{n}}(\mathbf{J}^0) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}^1}$$

and making a similar expansion for the perturbation H_1, explain how it is possible to choose the $S_{\mathbf{n}}(\mathbf{J}^0)$ to eliminate the $O(\epsilon)$ dependence of the Hamiltonian on the angles $\boldsymbol{\theta}^1$.

2. A satellite orbits an axisymmetric planet of mass M, whose potential may be expanded as $\Phi(r, \theta) = -GM/r + \Phi_1$, where

$$\Phi_1(r, \theta) = \frac{GM}{r} \sum_{l=2}^{\infty} \frac{J_l R^l}{r^l} P_l(\cos\theta). \tag{6.63}$$

Here R is the mean radius of the planet, the $P_l(\cos\theta)$ are Legendre polynomials and the J_l are multipole moments. Treating Φ_1 as a perturbation and considering only its $l = 2$ term, show using (2.26) or otherwise that the orbit-averaged perturbation Hamiltonian is given by

$$\langle H_1 \rangle = \frac{GMJ_2R^2}{4a^3(1-e^2)^{3/2}}(3\sin^2 i - 2), \tag{6.64}$$

where a, e and i are the semimajor axis, eccentricity and inclination of the satellite's orbit. By expressing this in terms of Delaunay variables, show that a, e and i are constants of motion, but that the argument of periapse ω undergoes prograde precession if $\cos i > 1/\sqrt{5}$, retrograde if $\cos i < 1/\sqrt{5}$.

3. Let us consider another way of deriving the equations of motion for the perturbed orbit-averaged Kepler problem. Introduce the dimensionless angular momentum

$$\mathbf{j} \equiv \frac{1}{\sqrt{GMa}}\mathbf{L}. \tag{6.65}$$

Directly from the usual properties of Poisson brackets it follows that the components of $\mathbf{j}$ satisfy

$$[j_i, j_j] = \frac{1}{\sqrt{GMa}}\epsilon_{ijk}j_k. \tag{6.66}$$

(a) Show that the components of the eccentricity vector (2.21) similarly satisfy

$$[e_i, e_j] = \frac{1}{\sqrt{GMa}}\epsilon_{ijk}j_k, \tag{6.67}$$

and that

$$[j_i, e_j] = \frac{1}{\sqrt{GMa}}\epsilon_{ijk}e_k. \tag{6.68}$$

(b) Now let $\bar{H}$ be a perturbation Hamiltonian that is independent of the mean anomaly. Show that under the influence of this $\bar{H}$ the vectors $\mathbf{j}$ and $\mathbf{e}$ evolve according to

$$\begin{aligned}\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} &= -\frac{1}{\sqrt{GMa}}\left(\mathbf{j}\times\frac{\partial\bar{H}}{\partial\mathbf{j}} + \mathbf{e}\times\frac{\partial\bar{H}}{\partial\mathbf{e}}\right),\\ \frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} &= -\frac{1}{\sqrt{GMa}}\left(\mathbf{e}\times\frac{\partial\bar{H}}{\partial\mathbf{j}} + \mathbf{j}\times\frac{\partial\bar{H}}{\partial\mathbf{e}}\right),\end{aligned} \tag{6.69}$$

which are known as the **Milankovich equations**.

4. In the notes above we investigated a couple of special cases of the restricted three-body problem. Let us now consider it more generally before looking at an interesting application. Using the notation introduced in §6.2 we can substitute $\mathbf{r} = a(\cos\varphi\hat{\mathbf{e}} + \sin\varphi\hat{\mathbf{u}})$ and $\mathbf{r}_\mathrm{p} = a_\mathrm{p}(\cos\varphi_\mathrm{p}\hat{\mathbf{e}}_\mathrm{p} + \sin\varphi_\mathrm{p}\hat{\mathbf{u}}_\mathrm{p})$ into the Hamiltonian (6.8) and – in principle – expand the result as a Fourier series in the Delaunay variables of planet and test particle.

(a) State why the semimajor axes a and a_p become constants of motion when we average the Hamiltonian over the mean anomalies w and w_p.

(b) After this averaging, the planet and test-particle orbits both become Keplerian ellipses, each of which we can characterise by its angular momentum and eccentricity vectors. Explain why the magnitude of the test particle's angular momentum vector changes much more slowly than its direction. How does the precession rate of the test particle's ellipse scale with m_p?

(c) Averaging over the precession period of each ellipse, the system becomes a pair of circular annuli, whose relative orientation evolves according to their averaged torque. Write down an order-of-magnitude estimate for this torque in terms of m_p and the characteristic radius R, assuming that $a \sim a_\mathrm{p} \sim R$. What sets the shortest timescale for which we can assume that this torque acts?

(d) Now let us apply this to a cluster of stars around a black hole: the central, dominant body becomes a black hole of mass $M_\bullet$, while the "planet" and test particle turn into stars that orbit this black hole. We suppose that there are such N stars, each of mass m, but with $Nm \ll M_\bullet$. Averaging the stars' orbits into circular annuli as above, write down an order-of-magnitude expression for the characteristic torque felt by each star in terms of G, N, m and R.

(e) Stating clearly any assumptions that you make, estimate how long it takes for the direction of a star's angular momentum vector to lose memory of its initial condition. Comment on how your result differs from the two-body relaxation timescale (3.7). See also Rauch & Tremaine (1996).

5. In §6.4 we skipped over the problem of calculating the constants C and B that appear in the resonant Hamiltonian (6.30) for the planar restricted three-body problem (6.19). This exercise rectifies that.

(a) Writing $r = a(1 - e\cos\eta)$ and $r_\mathrm{p} = a_\mathrm{p}(1 - e_\mathrm{p}\cos\eta_\mathrm{p})$, where η and η_p are the eccentric anomalies of the test particle and the planet, respectively, and Taylor expanding the square root in (6.19) in powers of $\alpha \equiv a/a_\mathrm{p}$, show that the perturbation Hamiltonian can be written as

$$\begin{aligned}\epsilon H_1 = -\frac{Gm_\mathrm{p}(1 - e_\mathrm{p}\cos\eta_\mathrm{p})}{a_\mathrm{p}}\Bigg[&\sum_{j=0}^{\infty} b_{1/2}^{(j)}(\alpha)\cos j\phi \\ &+ (\alpha - \cos\phi)\,\alpha\,(e\cos\eta - e_\mathrm{p}\cos\eta_\mathrm{p})\sum_{j=0}^{\infty} b_{3/2}^{(j)}(\alpha)\cos j\phi \\ &- \alpha(1 - e\cos\eta + e_\mathrm{p}\cos\eta_\mathrm{p})\cos\phi\Bigg] + O(e^2),\end{aligned} \tag{6.70}$$

where the **Laplace coefficients**

$$b_s^{(j)}(\alpha) \equiv \begin{cases} \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}\phi}{(1+\alpha^2-2\alpha\cos\phi)^s}, & j = 0, \\ \frac{2}{\pi}\int_0^\pi \frac{\cos j\phi\,\mathrm{d}\phi}{(1+\alpha^2-2\alpha\cos\phi)^s}, & j \neq 0, \end{cases} \tag{6.71}$$

come from the Fourier expansion $\left[1 + \alpha^2 - 2\alpha\cos\phi\right]^{-s} = \sum_{j=0}^{\infty} b_s^{(j)}\cos j\phi$.

(b) Now we need to rewrite the $\cos j\phi$ and $e\cos\eta$ factors that appear here in terms of the modified Delaunay angles (5.64) $\boldsymbol{\theta} = (\theta_\varpi, \theta_\Omega, \theta_\lambda) = (-\varpi, -\Omega, \lambda)$. Use Kepler's equation to show that

$$\begin{aligned}\cos w &= \cos\eta + e\sin^2\eta + O(e^2), \\ \sin w &= (1 - e\cos\eta)\sin\eta,\end{aligned} \tag{6.72}$$

and hence obtain an expression for $e\cos\eta$ in terms of λ and ϖ that is correct to first order in e.

(c) To deal with the $\cos j\phi$ factors, notice that the angle between the test body and planet is $\phi = (\varphi + \varpi) - (\varphi_\mathrm{p} + \varpi_\mathrm{p})$, where φ and φ_p are the respective true anomalies. Using (2.25), or otherwise, show that

$$\cos j(\varphi + \varpi) = \cos j\lambda - 2je\sin(\lambda - \varpi)\sin j\lambda + O(e^2), \tag{6.73}$$

$$\sin j(\varphi + \varpi) = \cos j\lambda + 2je\sin(\lambda - \varpi)\cos j\lambda + O(e^2). \tag{6.74}$$

(d) For simplicity let us assume that the planet is on a circular orbit so that $e_\mathrm{p} = 0$. Then we may set $\varpi_\mathrm{p} = 0$ and have $\eta_\mathrm{p} = \varphi_\mathrm{p} = \lambda_\mathrm{p}$. Show that

$$\cos j\phi = \cos j(\lambda - \lambda_\mathrm{p}) - 2je\sin(\lambda - \varpi)\sin j(\lambda - \lambda_\mathrm{p}) + O(e^2). \tag{6.75}$$

(e) By substituting these expressions for $\cos\eta$ and $\cos j\phi$ into (6.70) and comparing to the $K : K_\mathrm{p} = 3 : 2$ resonant perturbation Hamiltonian (6.23) obtain an expression for the constants β and $B(\mathbf{J})$ in the latter. Express your answer for $B(\mathbf{J})$ in terms of α, e and Laplace coefficients.

(f) Estimate the constant C that appears in the full resonant Hamiltonian (6.30) and comment on how well this resonant Hamiltonian describes the motion of the real Sun–Neptune–Pluto system.

Chapter 7

Galaxies: equilibrium models

We make the collisionless approximation (§3.1) for the motion of stars and dark matter. Then, assuming that stars are neither created nor destroyed, the phase-space distribution function (DF) $f(\mathbf{x}, \mathbf{x}, t)$ of each stellar (or DM) population satisfies the CBE (3.8),

$$\frac{\partial f}{\partial t} + [f, H] = 0, \tag{7.1}$$

in which $H = \frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{x}, t)$ is one-body Hamiltonian, with Φ the gravitational potential sourced by all matter in the system. If the galaxy has distinct stellar populations (e.g., different ages, metallicities) then we can either make this f a function of the stellar populations too $f(\mathbf{x}, \mathbf{v}, \tau, Z, ..., t)$, or we could introduce a distinct DF $f_1, ..., f_n$ for each stellar (or dark matter) population. These distinct populations are then coupled only by their effect on the overall potential Φ.

Note that in these lectures I define the DF f to be the phase-space *mass* density of one such population. We could equally well define it to be the number density, probability density or luminosity density of stars in phase space: for most purposes all are equivalent, save for some normalisation constants. We do not assume that this DF f is the only source of matter that contributes to the potential Φ, unless stated otherwise.

7.1 The Jeans and virial equations

The velocity moments of the DF are

$$\overline{\rho v_x^\alpha v_y^\beta v_z^\gamma}(\mathbf{x}, t) \equiv \int v_x^\alpha v_y^\beta v_z^\gamma \, f(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}^3\mathbf{v}, \tag{7.2}$$

in which $\alpha, \beta, \gamma \geq 0$ are integers. The zeroth-order moment ($\alpha = \beta = \gamma = 0$) is the mass density, $\rho(\mathbf{x}, t)$.

The first-order velocity moments are the density-weighted mean-streaming velocities,

$$\rho \bar{v}_i(\mathbf{x}, t) \equiv \int v_i \, f(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}^3\mathbf{v}, \tag{7.3}$$

for $i = x$, y or z. We can split the second-order moments

$$\begin{aligned} \rho \bar{v}_i v_j(\mathbf{x}, t) &\equiv \int v_i v_j \, f(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}^3\mathbf{v} \\ &= \rho \left(\bar{v}_i \bar{v}_j + \sigma_{ij} \right) \end{aligned} \tag{7.4}$$

into contributions from mean-streaming (first term) plus random motions, quantified by the velocity dispersion tensor (second term). The third- and higher-order moments do not have such direct physical interpretations.

7.1.1 Jeans equations

Multiplying the CBE

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} - \frac{\partial \Phi}{\partial x_j}\frac{\partial f}{\partial v_j} = 0 \tag{7.5}$$

by powers of v_i and integrating over velocities gives the sequence of **Jeans equations**, namely

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho\overline{v_i}) &= 0, \\ \frac{\partial}{\partial t}(\rho\overline{v_i}) + \frac{\partial}{\partial x_j}(\rho\overline{v_i v_j}) + \rho\frac{\partial \Phi}{\partial x_i} &= 0, \end{aligned} \tag{7.6}$$

and so on, where we have used integration by parts assuming that f vanishes sufficiently quickly as $v \to \infty$. These are sometimes called the equations of "stellar hydrodynamics": the first is simply the continuity equation; the second is the momentum equation. Writing $v_i\bar{v}_j = \bar{v}_i\bar{v}_j + \sigma_{ij}$ and using the first equation, the second can be written as

$$\rho\frac{\partial \bar{v}_i}{\partial t} + \rho\bar{v}_j\frac{\partial \bar{v}_i}{\partial v_j} + \frac{\partial}{\partial x_j}(\rho\sigma_{ij}^2) + \rho\frac{\partial \Phi}{\partial x_i} = 0, \tag{7.7}$$

which is Euler's equation for fluid flow with $\rho\sigma_{ij}^2$ playing the role of pressure. Unlike the situation in fluid mechanics, where, e.g., under certain circumstances we might reasonably assume that the gas is isothermal, in stellar dynamics we do not have an equation of state to tell us what to take for σ_{ij}^2. We can write down higher-order Jeans equations that express the rate of change of the n^{th}-order moments to those of order $n-1$ and $n+1$, but in the absence of some closure relation these are of limited use.

7.1.2 Virial equations

Multiplying the second Jeans equation by x_k and integrating over all space gives

$$\int \mathrm{d}^3\mathbf{x}\, x_k \frac{\partial}{\partial t}\rho\overline{v_i} + \int \mathrm{d}^3\mathbf{x}\, x_k \frac{\partial}{\partial x_j}\rho\overline{v_i v_j} + \int \mathrm{d}^3\mathbf{x}\, x_k \rho\frac{\partial \Phi}{\partial x_i} = 0. \tag{7.8}$$

Now take the time derivative outside the integral in the first term, and apply the divergence theorem to the second, making the reasonable assumption that the second moment vanishes sufficiently rapidly at infinity. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\mathbf{x}\,\rho x_k\overline{v_i} - \underbrace{\int \mathrm{d}^3\mathbf{x}\,\rho\overline{v_i v_k}}_{2K_{ik}} + \underbrace{\int \mathrm{d}^3\mathbf{x}\, x_k\rho\frac{\partial \Phi}{\partial x_i}}_{-W_{ik}} = 0, \tag{7.9}$$

where

$$\begin{aligned} K_{ij} &\equiv \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\rho\overline{v_i v_j}, \\ W_{ij} &\equiv -\int \mathrm{d}^3\mathbf{x}\, x_j\rho\frac{\partial \Phi}{\partial x_i}, \end{aligned} \tag{7.10}$$

are the usual kinetic and potential energy tensors. With the help of the continuity equation (7.6), the integral in the first term can be written as the time derivative of the moment of inertia tensor,

$$I_{ij} \equiv \int \mathrm{d}^3\mathbf{x}\,\rho x_i x_j. \tag{7.11}$$

Then swapping indices $i \leftrightarrow k$ in (7.9) and adding, we have the **virial equations**, or **tensor virial theorem**,

$$\frac{1}{2}\frac{d^2 I_{ik}}{\mathrm{d}t^2} = 2K_{ik} + W_{ik}. \tag{7.12}$$

The LHS of this vanishes for steady-state systems . Taking the trace gives

$$2K + W = 0, \tag{7.13}$$

which is known as the **scalar virial theorem**.

From N-body experiments, we know that galaxies settle down into approximate equilibrium ("virialise") within a few dynamical times. This suggests that that a good way of understanding them is to think of them as some underlying equilibrium $f(\mathbf{x}, \mathbf{v}), \Phi(\mathbf{x})$ plus perturbations on top of this.

7.1.3 Applications of moment equations

We have derived the Jeans and virial equations in Cartesian coordinates, but it is usually more natural to describe galaxies using either spherical polar or cylindrical polar coordinates. In spherical polars (r, ϑ, φ) the CBE $\partial f/\partial t + [f, H] = 0$ becomes

$$\begin{aligned}\frac{\partial f}{\partial t} + p_r \frac{\partial f}{\partial r} + \frac{1}{r^3}\left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta} - r^3\frac{\partial \Phi}{\partial r}\right)\frac{\partial f}{\partial v_r} + \frac{p_\vartheta}{r^2}\frac{\partial f}{\partial \vartheta} + \left(\frac{p_\vartheta^2 \cos\vartheta}{r^2\sin^3\vartheta} - \frac{\partial \Phi}{\partial \vartheta}\right)\frac{\partial f}{\partial p_\vartheta} \\ + \frac{p_\varphi}{r^2\sin^2\vartheta}\frac{\partial f}{\partial \varphi} - \frac{\partial \Phi}{\partial \varphi}\frac{\partial f}{\partial p_\varphi} = 0,\end{aligned} \tag{7.14}$$

with $p_r = \dot{r}$, $p_\vartheta = r^2\dot{\vartheta}$, $p_\varphi = r^2\sin^2\vartheta\dot{\varphi}$. For a steady-state, spherical non-rotating galaxy ($\partial_t = \partial_\vartheta = \partial_\varphi = \bar{v}_r = \bar{v}_\vartheta = \bar{v}_\varphi = 0$) it is natural to assume a velocity dispersion tensor $\sigma^2 = \mathrm{diag}(\sigma_r^2, \sigma_\vartheta^2, \sigma_\varphi^2)$ with $\sigma_\vartheta^2 = \sigma_\varphi^2$, but not necessarily equal to σ_r^2. The second-order steady-state Jeans equations then reduce to

$$\frac{\mathrm{d}}{\mathrm{d}r}\rho\sigma_r^2 + \frac{2\beta}{r}\rho\sigma_r^2 = -\rho\frac{\mathrm{d}\Phi}{\mathrm{d}r}, \tag{7.15}$$

where $\beta(r) \equiv 1 - \sigma_\varphi^2(r)/\sigma_r^2(r)$. This is the "equation of hydrostatic equilibrium" for spherical galaxies. Observations of distant galaxies provide (line-of-sight projections of) their luminosity-weighted $j(r)$ and dispersion $j(r)\sigma_r^2(r)$ profiles, but we can immediately rewrite this equation with $\rho(r)$ replaced by $j(r)$ to deal with this. [This corresponds to taking moments of the phase-space luminosity density instead of the mass density.] If we knew $\beta(r)$ then we could deduce the galaxy's acceleration profile $\mathrm{d}\Phi/\mathrm{d}r$ and therefore its mass distribution. It is tempting to assume that the velocity dispersion is isotropic ($\beta = 0$), but there is overwhelming evidence (e.g., from applications of the virial theorem or from more sophisticted modelling – see below) that this appealing assumption is not respected by real galaxies.

Another example comes from the dynamics of axisymmetric discs. One of the three second-order Jeans equations is

$$\frac{\partial}{\partial R}\rho\overline{v_R^2} + \frac{\partial}{\partial z}\rho\overline{v_R v_z} + \rho\left(\frac{\overline{v_R^2} - \overline{v_\varphi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0. \tag{7.16}$$

Multiply by R/ρ and focus on $z = 0$, where we assume that $\partial\rho/\partial z = 0$:

$$\frac{R}{\rho}\frac{\partial}{\partial R}\rho\overline{v_R^2} + R\frac{\partial}{\partial z}\overline{v_R v_z} + \overline{v_R^2} - \overline{v_\varphi^2} + R\frac{\partial \Phi}{\partial R} = 0. \tag{7.17}$$

Now split $\overline{v_\varphi^2} = \sigma_\varphi^2 + \bar{v}_\varphi^2$ (velocity dispersion plus mean-streaming) and split $\bar{v}_\varphi$ further into

$$\bar{v}_\varphi = v_\mathrm{c} - v_\mathrm{a}, \tag{7.18}$$

where $v_\mathrm{c}^2 \equiv R\partial\Phi/\partial R$ is the local circular speed and v_a, the **asymmetric drift**, is the amount by which $\bar{v}_\varphi$ lags behind v_c. Rearranging, we obtain

$$\begin{aligned}\sigma_\varphi^2 - \overline{v_R^2} - \frac{R}{\rho}\frac{\partial}{\partial R}\rho\overline{v_R^2} - R\frac{\partial}{\partial z}\overline{v_R v_z} &= v_\mathrm{c}^2 - \bar{v}_\varphi^2 \\ &= v_\mathrm{a}(2v_\mathrm{c} - v_\mathrm{a}) \simeq 2v_\mathrm{c}v_\mathrm{a}.\end{aligned} \tag{7.19}$$

That is, the larger the velocity disperion σ_φ of a stellar population, the more it is supported by "pressure" and the more its mean rotational velocity $\bar{v}_\varphi$ lags behind the local circular speed v_c.

7.2 Jeans' theorem

The Jeans and virial equations are useful aids for understanding the gross dynamics of galaxies, but the lack of any closure relations limits their predictive power. To understand the internal dynamics of galaxies in equilibrium more deeply we need to engage with the orbits that their potentials $\Phi(\mathbf{x}, t)$ can support. The most important consequence is **Jeans' theorem**: the DF of an equilibrium galaxy ($\partial f/\partial t = 0$) can depend on $(\mathbf{x}, \mathbf{v})$ only through integrals of motion. That is, $f = f(I_1(\mathbf{r}, \mathbf{v}), I_2(\mathbf{r}, \mathbf{v}), I_3(\mathbf{r}, \mathbf{v}))$, where the I_i are the integrals of motion. For example, in an axisymmetric galaxy the integrals of motion I_i could be energy E, the z component of angular momentum L_z and the nonclassical third integral I_3.

The **strong Jeans' theorem** states that the DF of a steady-state galaxy in which almost all orbits are regular and nonresonant is a function only of the actions $\mathbf{J}$. To show this we first note that, by assumption, angle–action variables exist for the potential Φ. Suppose that at time $t = 0$ the DF is

$$f(\boldsymbol{\theta}, \mathbf{J}, t = 0) = \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{J}) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}. \tag{7.20}$$

Since $\boldsymbol{\theta}(t) = \boldsymbol{\Omega}(\mathbf{J})t + \boldsymbol{\theta}(0)$ and the value of f is unchanged under the $\boldsymbol{\theta}$ flow, it follows that at a later time t

$$f(\boldsymbol{\theta}, \mathbf{J}, t) = \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{J}) \mathrm{e}^{\mathrm{i}(\boldsymbol{\theta} - \mathbf{n}\cdot\boldsymbol{\Omega}t)}. \tag{7.21}$$

Differentiating w.r.t. t,

$$\frac{\partial f}{\partial t} = -\mathbf{i} \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{J}) \mathbf{n}\cdot\boldsymbol{\Omega} \mathrm{e}^{\mathrm{i}(\boldsymbol{\theta} - \mathbf{n}\cdot\boldsymbol{\Omega}t)}, \tag{7.22}$$

which, when the potential is nonresonant ($\mathbf{n}\cdot\boldsymbol{\Omega} \neq 0$), is zero only if $f_{\mathbf{n}\neq\mathbf{0}} = 0$.

7.3 Simple equilibrium galaxy models

Now let us use Jeans' theorem to construct some simple galaxy models. We consider only the simplest examples, which are expressed as functions of the standard integrals of motion (e.g., energy and angular momentum) instead of actions. To make things even easier we consider only the simplest scale-free forms for the potential, a consequence of which is that they have infinite mass.

The **Mestel disc** is a razor-thin axisymmetric disc with DF

$$f(E, L_z) = \begin{cases} f_0 \left(\frac{L_z}{R_0 v_\mathrm{c}}\right)^q \mathrm{e}^{-E/\sigma^2}, & L_z > 0, \\ 0, & L_z < 0, \end{cases} \tag{7.23}$$

in potential $\Phi(R) = v_\mathrm{c}^2 \log(R/R_0)$, where v_c, q and σ are free parameters.

Self-consistency requires that

$$q = \frac{v_\mathrm{c}^2}{\sigma^2} - 1, \quad f_0 = \frac{\Sigma_0 v_\mathrm{c}^q}{2^{q/2}\sqrt{\pi}\Gamma\left(\frac{q+1}{2}\right)\sigma^{2+q}}, \tag{7.24}$$

where Σ_0 is the characteristic surface density. It is not hard to show that $\overline{v_R^2} = \sigma^2$ and that the larger q is the "colder" the disc becomes in v_φ.

As an example of a spherically symmetric model, let's take the isothermal sphere, for which the DF

$$f(E) = \frac{\rho_0}{(2\pi)^{3/2}\sigma^3} \exp[-E/\sigma^2] = \frac{\rho_0}{(2\pi)^{3/2}\sigma^3} \exp\left[-\frac{\Phi(r) + \frac{1}{2}v^2}{\sigma^2}\right]. \tag{7.25}$$

The density is

$$\rho(r) = \int f \mathrm{d}^3\mathbf{v} = \rho_0 \exp[-\Phi(r)/\sigma^2]. \tag{7.26}$$

The velocity dispersion tensor is isotropic with dispersion σ. We can demand that this be consistent with Poisson's equation,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho(r) = 4\pi G\rho_0\exp[-\Phi(r)/\sigma^2]. \tag{7.27}$$

We can solve this numerically by marching outwards from $r = 0$ with assumed initial values of $\Phi(0)$ and $\mathrm{d}\Phi/\mathrm{d}r$. The only known analytical solution is the **singular isothermal sphere**, for which $\Phi = 2\sigma^2\log r$ and density $\rho \propto 1/r^2$. All self-consistent solutions for the DF (7.25) have infinite mass, $\rho \sim r^{-2}$ as $r \to \infty$.

One can use the the same broad idea to deal with more general DFs of the form $f(\mathbf{J})$.

Exercises

1. (a) For a razor-thin disc having DF $f(E, L_z)$ in potential $\Phi(R)$ show that the velocity-space volume element at radius R can be written as

$$\mathrm{d}v_R\mathrm{d}v_\varphi = \frac{2\mathrm{d}E\mathrm{d}L_z}{[2R^2(E-\Phi(R)) - L_z^2]^{1/2}}. \tag{7.28}$$

Explain the origin of the factor of 2 that appears in the numerator of this expression. The energy E satisfies $E > \Phi(R)$. What are the corresponding bounds on L_z?
(b) Using this, or otherwise, show that the velocity moments of the Mestel DF (7.23) are given by

$$\overline{\rho v_R^\alpha v_\phi^\beta}(R) = \frac{2^{\frac{\alpha+\beta+q}{2}}\sigma^{2+\alpha+\beta+q}}{v_\mathrm{c}^q}\left(\frac{R}{R_0}\right)^q f_0\Gamma\left(\frac{q+\beta+1}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)\mathrm{e}^{-\Phi(R)/\sigma^2}. \tag{7.29}$$

Hence obtain the constraints (7.24).

Chapter 8

Galaxies: perturbations

8.1 N-body simulation

The most powerful way of probing how a galaxy model responds to perturbations is by using N-body simulations. There are two broad types of N-body simulation in galaxy dynamics: collisional and collisionless. Collisional or "direct" N-body models are what you might immediately think of when you hear "N-body simulation": they are direct simulations in which each particle in a physical star (or planet or whatever). They are often used in simulations of star clusters to study relaxation, evaporation and similar processes. Because each particle represents a star one can even include simple models of stellar evolution and include detailed models for what happens when two stars pass close enough to raise tides or even collide.

Collisionless N-body simulations are a little more subtle (Leeuwin et al., 1993). The CBE,

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0, \tag{8.1}$$

is a first-order quasilinear PDE, which suggests we try solving it using the method of characteristics (e.g., Arnold, 1992). Its characteristic curves are simply

$$\mathrm{d}t = \frac{\mathrm{d}x_i}{v_i} = \frac{\mathrm{d}v_i}{-\partial\Phi/\partial x_i}, \tag{8.2}$$

which, when written out, are no more than the equations of motion for a test particle moving in the potential Φ. The only difficulty is that Φ itself is set by f: we have that

$$\Phi(\mathbf{x}) = -GM \int \frac{f(\mathbf{x}', \mathbf{v}')\,\mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{v}'}{|\mathbf{x} - \mathbf{x}'|} + \Phi_{\text{external}}. \tag{8.3}$$

In a collisionless simulation the particles are simply a Monte Carlo realisation of the underlying DF, which are evolved forward in time by following the characteristics of the CBE in the (Monte Carlo-sampled) potential. That's the general idea. In practice, a collisionless code code has two main parts: (i) a potential solver, which is used to estimate Φ and its derivatives from the discrete realisation of f, somehow accounting for the integrable singularity in the integrand in (8.3); (ii) the integrator, which is used to advance the particles forward in time (8.2). In practice the integrator is nearly always based on some variant of the leapfrog scheme, but there are many choices for the Poisson solver, including mesh-based schemes, truncated basis function expansions, tree expansions, all of which have their pros and cons depending on the problem at hand.

8.2 Linear perturbation theory

Powerful as they are, it is usually quite difficult to extract insight from the output of N-body simulations. A natural alternative then is to consider the linear response of galaxies to internal or external perturbations.

Take a collisionless galaxy model with an equilibrium Hamiltonian H and DF $F(\mathbf{w})$. Apply an external perturbation $\epsilon h^{\rm e}(\mathbf{w},t)$, in response to which the stars' DF changes to $F(\mathbf{w})+\epsilon f(\mathbf{w},t)$. Let $h^f(\mathbf{w},t)$ be the change in the Hamiltonian caused by this response: usually this will be

$$h^f(\mathbf{x},\mathbf{v},t) = -G\int \frac{f(\mathbf{x}',\mathbf{v}',t)\,{\rm d}^3\mathbf{x}'{\rm d}^3\mathbf{v}'}{|\mathbf{x}-\mathbf{x}'|}. \tag{8.4}$$

The overall Hamiltonian is then $H+\epsilon h$, with $h=h^{\rm e}+h^f$ and the CBE of the perturbed system becomes

$$\epsilon\left(\frac{\partial f}{\partial t}+[f,H]+[F,h]\right)+\epsilon^2[f,h]=0. \tag{8.5}$$

Here our ambition is limited to working out the linearized response. So we throw away the nonlinear $\epsilon^2[f,h]$ and write what's left as

$$\left.\frac{{\rm d}f}{{\rm d}t}\right|_H \equiv \frac{\partial f}{\partial t}+[f,H]=-[F,h], \tag{8.6}$$

the LHS of which means that the derivative of f is taken along orbits in the unperturbed Hamiltonian H.

Integrating this with respect to time under the assumption that the perturbation f was zero in the distant past, the linearized response satisifes

$$f(\mathbf{w}_H(t),t) = -\int_{-\infty}^{t}{\rm d}t'\,[F,h](\mathbf{w}_H(t')), \tag{8.7}$$

where $\mathbf{w}_H(t)$ is the orbit of any star in the the unperturbed H. To distinguish different orbits, we make this notation more explicit by writing $\mathbf{w}_H(t|\mathbf{w}',t')$ for the time-t phase-space location of an orbit that at time t' was at location $\mathbf{w}'$. Then a more complete expression for the response DF is

$$f(\mathbf{w},t) = -\int_{-\infty}^{t}{\rm d}t'\int{\rm d}\mathbf{w}'\,\delta\left(\mathbf{w}-\mathbf{w}_H(t|\mathbf{w}',t')\right)[F,h](\mathbf{w}'), \tag{8.8}$$

in which the Dirac delta selects only that $\mathbf{w}'$ at time t' that arrives at $\mathbf{w}$ at time t.

8.2.1 Matrix method

Equation (8.8) is not an explicit expression for f, however, because the integrand in the RHS depends $h=h^{\rm e}+h^f$ and h^f depends on f. Kalnajs (1976) addressed this problem by introducing potential–density pairs. We follow his idea by introducing potential–DF pairs $(h^\mu(\mathbf{w}),f^\mu(\mathbf{w}))$, the elements of each pair related via

$$h^\mu(\mathbf{w}) = \int{\rm d}^6\mathbf{w}'U(\mathbf{w},\mathbf{w}')f^\mu(\mathbf{w}'), \tag{8.9}$$

where $U(\mathbf{w},\mathbf{w}')$ is the two-body interaction Hamiltonian. For example, if F and f represent stellar densities then this will usually be $U(\mathbf{r},\mathbf{r}')=-G/|\mathbf{r}-\mathbf{r}'|$, but for comparison with N-body simulation it might be appropriate to use a softened version of this interaction kernel. When studying resonant relaxation around a black hole we might decide to average orbits over mean anomaly, in which case $U(\mathbf{w},\mathbf{w}')$ would be the wire–wire interaction Hamiltonian. We assume that U is real-valued and symmetric in its arguments.

Having chosen $f^\mu(\mathbf{w})$ and $U(\mathbf{w},\mathbf{w}')$ we can immediately write down a biorthogonality relation

$$\int[h^\mu(\mathbf{w})]^\star f^\nu(\mathbf{w})\,{\rm d}^6\mathbf{w}\equiv-\mathcal{E}^{\mu\nu}, \tag{8.10}$$

which uses $U(\mathbf{w},\mathbf{w}')$ to define a scalar product-like relation between different elements (f^μ,f^ν). (It is not a "good" scalar product though because it is not positive definite.)

We assume that our set of $h^\mu(\mathbf{w})$ is complete enough that we can expand the stimulus and response Hamiltonians as

$$\begin{aligned} h^{\mathrm{e}}(\mathbf{w},t) &= \sum_\mu a_\mu(t) h^\mu(\mathbf{w}), \\ h^f(\mathbf{w},t) &= \sum_\mu b_\mu(t) h^\mu(\mathbf{w}). \end{aligned} \tag{8.11}$$

Then one of the many possible response DFs that produces this $h^f(\mathbf{w},t)$ is simply

$$f(\mathbf{w},t) = \sum_\mu b_\mu(t) f^\mu(\mathbf{w}). \tag{8.12}$$

We can relate b's to a's by multiplying the integral expression (8.8) for $f(\mathbf{w},t)$ by $h^{\mu\star}$, integrating over $\mathbf{w}$ and using the orthogonality relation (8.10). The result is that

$$\mathcal{E}^{\mu\nu} b_\nu(t) = \int_{-\infty}^{t} \mathrm{d}t' K^{\mu\nu}(t,t') \left(a_\nu(t') + b_\nu(t')\right), \tag{8.13}$$

in which the kernel

$$\begin{aligned} K^{\mu\nu}(t,t') &\equiv \int \mathrm{d}^6\mathbf{w} \left(h^\mu(\mathbf{w})\right)^\star \int \mathrm{d}^6\mathbf{w}' [F,h^\nu](\mathbf{w}') \delta\left(\mathbf{w} - \mathbf{w}_H(t|\mathbf{w}',t')\right) \\ &= \int \mathrm{d}^6\mathbf{w}' \ \left(h^\mu(\mathbf{w}_H(t|\mathbf{w}',t'))\right)^\star [F,h^\nu](\mathbf{w}'), \end{aligned} \tag{8.14}$$

The second line makes it clear that $K^{\mu\nu}(t,t')$ quantifies how much an element of basis function h^ν introduced at time t' contributes to the element h^μ at a later time t. If H is autonomous (i.e., $\partial H/\partial t = 0$) then $\mathbf{w}_H(t|\mathbf{w}',t')$ is a map $\mathbf{w}_H(t-t'|\mathbf{w}')$ that depends only on the time difference $t-t'$ and the kernel simplifies to

$$K^{\mu\nu}(t,t') = K^{\mu\nu}(t-t') = \int \mathrm{d}^6\mathbf{w}' \ \left(h^\mu(\mathbf{w}_H(t-t'|\mathbf{w}'))\right)^\star [F,h^\nu](\mathbf{w}'). \tag{8.15}$$

Equation (8.13) is a linear Volterra equation for the density response $b_\mu(t)$ expressed as convolution with the kernel (8.15) of the stimulus $a_\mu(t)$ and response $b_\mu(t)$ at earlier times. Once a basis has been chosen and the kernel calculated, it is very easy to solve (8.13) numerically to find the linear response to any stimulus. Then, knowing $h(\mathbf{w},t)$ one can go back to (8.8) to find the full DF response.

8.2.2 The kernel in angle–action variables

Suppose that our unperturbed H is integrable with angle–action variables $(\boldsymbol{\theta},\mathbf{J})$. Writing $\mathbf{w}' = (\boldsymbol{\theta},\mathbf{J})$ our time-evolution map is simply

$$\mathbf{w}_H(t-t'|\mathbf{w}') = (\boldsymbol{\theta} + \boldsymbol{\Omega}(t-t'), \mathbf{J}), \tag{8.16}$$

where, as usual, $\boldsymbol{\Omega} \equiv \partial H/\partial \mathbf{J}$. Now we can immediately write down epressions for the various factors that appear in the kernel (8.15). The first factor in the integrand involves

$$h^\mu(w_H(t-t'|\mathbf{w}')) = \sum_{\mathbf{m}} h^\mu_{\mathbf{m}}(\mathbf{J}') \mathrm{e}^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\Omega}(t-t')} \mathrm{e}^{\mathrm{i}\mathbf{m}\cdot\theta}, \tag{8.17}$$

and the Poisson bracket is given by

$$[F,h^\nu] = -\sum_{\mathbf{n}} \mathrm{i}\left(\mathbf{n}\cdot\frac{\partial F}{\partial \mathbf{J}}\right) h_{\mathbf{n}}(\mathbf{J}) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}. \tag{8.18}$$

Integrating $\boldsymbol{\theta}$, the kernel itself becomes

$$K^{\mu\nu}(t-t') = -(2\pi)^3 \mathrm{i} \int \sum_{\mathbf{n}} \mathbf{n}\cdot\frac{\partial F}{\partial \mathbf{J}} \mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}(t-t')} (h^\mu_{\mathbf{n}}(\mathbf{J}))^\star h^\nu_{\mathbf{n}}(\mathbf{J}) \, \mathrm{d}^3\mathbf{J}. \tag{8.19}$$

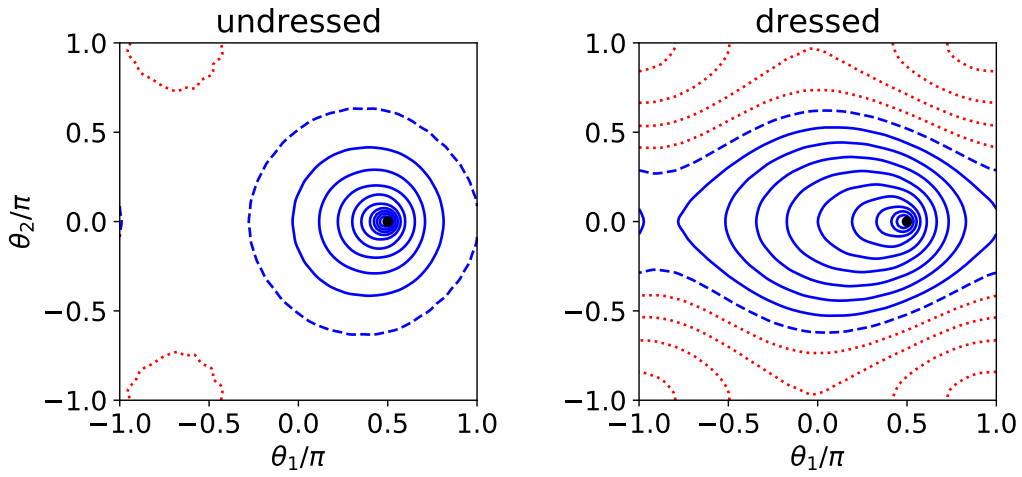

Figure 8.1: Undressed (left) and dressed (right) projected density response of a cube of mass $0.8M_\mathrm{J}$ with a Maxwell–Boltzmann DF having unit dispersion to a satellite dragged through the cube with constant velocity $\mathbf{J}_\mathrm{p} = (0.3, 0, 0)$. Contours are linearly spaced with the same levels on both panels. Positive densities are solid blue, negative dotted red. The dashed blue contour is the zero-(projected) density response.

8.3 Example: the periodic cube

The periodic cube is an extremely simple and completely unrealistic model of a galaxy, that is nevertheless useful for illustrating the application of perturbation theory to stellar dynamics (Magorrian, 2021). It treats the galaxy as a big box of stars. Stars are confined by making the box periodic: if a star passes through one wall of the box it immediately emerges on the opposite wall. We map $(\mathbf{x}, \mathbf{v}) \to (\boldsymbol{\theta}, \mathbf{J})$ so that the new coordinates $\boldsymbol{\theta}$ are 2π periodic.

Stars interact with some translation-invariant potential,

$$U(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = m_1 m_2 \sum_{\mathbf{n}} U_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)}, \tag{8.20}$$

that is completely described by the coefficients $U_{\mathbf{n}}$. Taking $U_{\mathbf{n}} = -4\pi G/\mathbf{n}^2$ for $\mathbf{n} \neq \mathbf{0}$ gives this produces familiar Newtonian gravity for separations $|\Delta\boldsymbol{\theta}| \ll 1$ small compared to the scale of the box. Independent of the choice of $U_{\mathbf{n}}$ the simplest equilibrium models have uniform density $\rho = \rho(\boldsymbol{\theta}) = \text{const}$, for which $H(\mathbf{J}) = \frac{1}{2}\mathbf{J}^2$ and we may choose any non-negative function $F(\mathbf{J})$ as an equilibrium DF.

For basis functions we take $h^\mu = h^{\mathbf{m}}(\boldsymbol{\theta}) = \mathrm{e}^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}}$, with the basis labels μ and ν being integer triplets $\mathbf{m}$ and $\mathbf{n}$. Then the orthogonality relation (8.10) gives $\mathcal{E}^{\mathbf{mn}} = -(2\pi)^3\delta_{\mathbf{mn}}/\mathbf{n}^2 U_{\mathbf{n}}$ and the Poisson bracket in the kernel becomes $[F, h^{\mathbf{n}}] = -\mathrm{i}(\mathbf{n}\cdot\partial F/\partial\mathbf{J})h^{\mathbf{n}}$. The kernel itself is

$$K^{\mathbf{mn}}(\Delta t) = -\mathrm{i}(2\pi)^3\delta_{\mathbf{mn}} \int \mathbf{n}\cdot\frac{\partial F}{\partial \mathbf{J}} \mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}\Delta t}\, \mathrm{d}^3\mathbf{J}. \tag{8.21}$$

8.3.1 Dynamical friction

Now let us consider a cube through which a satellite of mass M_p is passing through with velocity $\mathbf{J}_\mathrm{p}$. The coefficients representating its potential are then potential is then

$$a_{\mathbf{n}}(t) = \frac{M_\mathrm{p} U_{\mathbf{n}}}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\mathbf{J}_\mathrm{p} t}. \tag{8.22}$$

⋮

8.4 Another example: the shearing sheet

The shearing sheet (Julian & Toomre, 1965; Binney, 2020) is an idealised model of the local dynamics of a differentially rotating stellar disc. We zoom in on a patch of the disc and introduce a Cartesian coordinate system whose x axis is directed in the outward radial direction and whose y axis is in the direction of rotation. Binney (2020) shows that, making the epicycle approximation, a Hamiltonian for this system is

$$\begin{aligned} H(x,y,p_x,\Delta_y) &= \frac{1}{2}p_x^2 + \frac{1}{2}\Delta_y^2 - 2x\Omega\Delta_y + \frac{1}{2}\kappa^2 x^2 \\ &= H_x(x,p_x,\Delta_y) + H_y(\Delta_y), \\ H_x(x,p_x,\Delta_y) &= \frac{1}{2}\left[p_x^2 + \kappa^2(x-\bar{x})^2\right], \\ H_y(\Delta_y) &= -\frac{1}{2}\left(1 - \frac{2\Omega^2}{\kappa^2}\right)\Delta_y^2, \end{aligned} \tag{8.23}$$

where κ and Ω are the familiar epicycle frequencies and $\bar{x} \equiv 2\Omega\Delta_y/\kappa^2$.

For our equilibrium DF we adopt the Maxwellian

$$\begin{aligned} F(x,p_x,\Delta_y) &= \frac{\Omega\Sigma_0}{\pi\kappa\sigma^2}\exp[-H_x/\sigma^2] \\ &= \frac{\Omega\Sigma_0}{\pi\kappa\sigma^2}\exp\left[-\frac{p_x^2+\kappa^2(x-\bar{x})^2}{2\sigma^2}\right]. \end{aligned} \tag{8.24}$$

Using $\mathrm{d}^2\mathbf{v} = \mathrm{d}p_x\mathrm{d}\Delta_y = (\kappa^2/2\Omega)\mathrm{d}p_x\mathrm{d}\bar{x}$ it's easy to see that the equilibrium density of this model of the shearing sheet is Σ_0 and its velocity dispersions are $\sigma_x = \sigma$ and $\sigma_y = \frac{\kappa}{2\Omega}\sigma$.

8.4.1 Basis

The natural basis to choose to represent perturbations to the potential is

$$h^{\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}. \tag{8.25}$$

The lack of periodicity in the system means that $\mathbf{k} = (k_x, k_y)$ is a continuous variable and so the sums over ν in the Volterra equation (8.13) are replaced by integrals over $\mathbf{k}$. The corresponding density is

$$\Sigma^{\mathbf{k}}(\mathbf{x}) = -\frac{|\mathbf{k}|}{2\pi G}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{8.26}$$

and the scalar product-like relation (8.10) between $h^{\mathbf{k}}$ and $\Sigma^{\mathbf{k}'}$ becomes

$$\mathcal{E}^{\mathbf{k}\mathbf{k}'} = \frac{2\pi|\mathbf{k}|}{G}\delta(\mathbf{k}-\mathbf{k}'). \tag{8.27}$$

The Dirac delta in (8.27) means that we can integrate the Volterra equation (8.13) for this system over $\mu = \mathbf{k}'$ and use (8.26) to obtain an equivalent equation for the evolution of the density response in place of the potential. Expanding the density response as

$$\Sigma^f(\mathbf{x},t) = \int \mathrm{d}\mathbf{k}\,\tilde{\Sigma}^f_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{8.28}$$

the time-dependent expansion coefficients $\tilde{\Sigma}^f_{\mathbf{k}}(t)$ evolve according to

$$\tilde{\Sigma}^f_{\mathbf{k}}(t) = \int \frac{\mathrm{d}\mathbf{k}'\,G}{2\pi|\mathbf{k}'|}\int_{-\infty}^{t}\mathrm{d}t'\,K^{\mathbf{k}\mathbf{k}'}(t-t')\left(\tilde{\Sigma}^{\mathrm{e}}_{\mathbf{k}}(t') + \tilde{\Sigma}^f_{\mathbf{k}}(t')\right), \tag{8.29}$$

where the $\tilde{\Sigma}^{\mathrm{e}}_{\mathbf{k}}(t)$ are the expansion coefficients of an imposed internal matter distribution $\Sigma^{\mathrm{e}}(\mathbf{x},t)$. Alternatively, we can view them as resulting from an external stirring potential $\Phi^{\mathrm{e}}(\mathbf{x},t)$ through

$$\tilde{\Sigma}^{\mathrm{e}}_{\mathbf{k}}(t) = -\frac{|\mathbf{k}|}{(2\pi)^3 G}\int \mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\Phi^{\mathrm{e}}(\mathbf{x},t). \tag{8.30}$$

8.4.2 Kernel

With these choices of potential/density basis and equilibrium DF, the Poisson bracket that appears in the kernel (8.15) is given by

$$\begin{aligned}[F, h^{\mathbf{k}'}] &= -\mathrm{i}\mathbf{k}' \cdot \frac{\partial F}{\partial \mathbf{p}} h_{\mathbf{k}'} = \frac{\mathrm{i}}{\sigma^2}\mathbf{k}' \cdot \frac{\partial H_x}{\partial \mathbf{p}} F h^{\mathbf{k}'} \\ &= \frac{\mathrm{i}}{\sigma^2}\left(k'_x p_x - 2\Omega k'_y (x - \bar{x})\right) F h^{\mathbf{k}'}.\end{aligned} \tag{8.31}$$

and the expression for the kernel becomes

$$K^{\mathbf{k}\mathbf{k}'}(t - t') = \frac{\mathrm{i}}{\sigma^2}\int \mathrm{d}^4\mathbf{w}\, \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}(t) - \mathbf{k}'\cdot\mathbf{x}(t'))}\left(k'_x p_x(t') - 2\Omega k'_y (x(t') - \bar{x})\right) F(\mathbf{w}). \tag{8.32}$$

To make the connection with Binney (2020) transparent we choose to perform the integration over stars' phase-space locations at $t = 0$. Instead of using $\mathbf{w} = (x(0), y(0), p_x(0), \Delta_y)$ directly, however, we replace $x(t)$ by

$$x'(t) \equiv x(t) - \bar{x} \tag{8.33}$$

and Δ_y by $\bar{x}$. Then $\mathrm{d}^4\mathbf{w} = \frac{\kappa^2}{2\Omega}\mathrm{d}x'(0)\mathrm{d}y(0)\mathrm{d}p_x(0)\mathrm{d}\bar{x}$ and $F(\mathbf{w})$ becomes a product of Gaussians in p_x and x'. We obtain $\mathbf{x}(t)$ and $p_x(t)$ used in (8.32) by integrating the Hamiltonian (8.23) for time t from these initial conditions. The result is

$$\begin{aligned}x(t) &= \bar{x} + x'(0)\cos\kappa t + \frac{1}{\kappa}p_x(0)\sin\kappa t, \\ y(t) &= y(0) - 2At - \frac{2\Omega}{\kappa}\left[x'(0)\sin\kappa t + \frac{1}{\kappa}p_x(0)(1 - \cos\kappa t)\right], \\ p_x(t) &= p_x(0)\cos\kappa t - \kappa x'(0)\sin\kappa t,\end{aligned} \tag{8.34}$$

where the Oort constant $A \equiv \Omega(1 - \kappa^2/4\Omega^2)$.

Now all that remains is to carry out the integral (8.32). From equations (8.34) the factor in the middle of the integrand is a linear combination of $p_x(0)$ and $\kappa x'(0)$, namely

$$k'_x p_x(t') - 2\Omega k'_y x'(t') = C_p p_x(0) + C_{\kappa x}\kappa x'(0), \tag{8.35}$$

where we have introduced a vector $\mathbf{C}(\mathbf{k}', t') = (C_p, C_{\kappa x})$ with components

$$\begin{aligned}C_p(\mathbf{k}', t') &= k'_x C' - \frac{2\Omega}{\kappa}k'_y S', \\ C_{\kappa x}(\mathbf{k}', t') &= -k'_x S' - \frac{2\Omega}{\kappa}k'_y C',\end{aligned} \tag{8.36}$$

with $C' = \cos\kappa t$ and $S' = \sin\kappa t$. Similarly, the argument of the complex exponential is a linear combination of all four $t = 0$ phase-space coordinates:

$$\begin{aligned}\mathbf{k}\cdot\mathbf{x}(t) - \mathbf{k}'\cdot\mathbf{x}(t') = B_{\kappa x}\kappa x'(0) + (k_y - k'_y)y(0) + B_p p_x(0) \\ + \left(k_x - k'_x + 2A(k_y t - k'_y t')\right)\bar{x},\end{aligned} \tag{8.37}$$

in which $\mathbf{B}(\mathbf{k}, \mathbf{k}', t, t') = (B_p, B_{\kappa x})$ is another two-dimensonal vector with components

$$\begin{aligned}B_x(\mathbf{k}, \mathbf{k}', t, t') &= k_x C - k'_x C' - \frac{2\Omega}{\kappa}(k_y S - k'_y S'), \\ B_p(\mathbf{k}, \mathbf{k}', t, t') &= k_x S - k'_x S' + \frac{2\Omega}{\kappa}(k_y C - k'_y C'),\end{aligned} \tag{8.38}$$

with $C = \cos\kappa t$, $S = \sin\kappa t$. Then, using standard Fourier integrals

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx} &= 2\pi\delta(k), \\ \int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}(a + bx)\mathrm{e}^{-x^2/2\sigma^2} &= \sqrt{2\pi}\sigma(a + \mathrm{i}bk\sigma)\mathrm{e}^{-k^2\sigma^2/2},\end{aligned} \tag{8.39}$$

or by turning to Appendix A of Binney (2020), the kernel (8.32) becomes

$$\begin{aligned} K^{\mathbf{k}\mathbf{k}'}(t-t') &= (2\pi)^2\Sigma_0\delta(k_y-k_y')\delta\left(k_x-k_x'+2Ak_y(t-t')\right) \\ &\times \mathbf{B}\cdot\mathbf{C}\exp\left[-\frac{1}{2}\sigma^2\mathbf{B}\cdot\mathbf{B}\right]. \end{aligned} \tag{8.40}$$

As expected, the kernel depends on t and t' only through their difference $t-t'$: the expression on the right-hand side obscures this, but notice that adding some Δt to both t and t' rotates the vectors $\mathbf{B}$ and $\mathbf{C}$ by the same angle $\kappa\Delta t$, leaving the scalar products $\mathbf{B}\cdot\mathbf{C}$ and $\mathbf{B}\cdot\mathbf{B}$ unchanged.

The Dirac deltas in this expression mean that a perturbation with wavenumber $\mathbf{k}(t') = (k_x', k_y')$ at time t' evolves after time $t-t'$ to another wavenumber $\mathbf{k}(t) = (k_x' + 2Ak_y(t-t'), k_y')$, with a new amplitude. This is good news, because it shows that the kernel is diagonal, albeit not on the *main* diagonal. It couples basis vectors $h^{\mathbf{k}}$, $h^{\mathbf{k}'}$ in an awkward, time-dependent way.

8.4.3 Swing amplification

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Appendix A

Orbit integration code

Here is the implementation of a simple leapfrog code that was used to generate the orbits plotted in Figure 3.3. It is also available here.

```
#!/usr/bin/env python3
# Examine orbits in rotating flattened logarithmic potential

import numpy as np
import matplotlib.pyplot as plt

class RotLogPot:
    def __init__(self):
        self.q = 0.8
        self.Rc = 0.1
        self.Omegap = 1.0
    def Phi(self,x):
        return 0.5*np.log(self.Rc**2+x[0]**2+x[1]**2/self.q**2)
    def dPhidx(self,x):
        dnom = self.Rc**2+x[0]**2+x[1]**2/self.q**2
        return np.array([x[0],x[1]/self.q**2])/dnom
    def HJ(self,x,p):
        "return the value of the Jacobi integral"
        return p.dot(p)/2 - self.Omegap*(x[0]*p[1]-x[1]*p[0]) + self.Phi(x)
    def drift(self,x,p,dt):
        "Carry out a drift leapfrog step."
        alpha = -dt*self.Omegap # This is the angle by which frame rotates in time dt
        rotmat = np.array([[ np.cos(alpha), -np.sin(alpha)],
                           [np.sin(alpha), np.cos(alpha) ]])
        return np.dot(rotmat,x+dt*p), np.dot(rotmat,p)
    def kick(self,x,p,dt):
        "Carry out a kick leapfrog step"
        return x,p-self.dPhidx(x)*dt

def stepit(pot,x,p,dt=0.01,nstep=10000):
    ans = []
    for istep in range(nstep):
        x,p = pot.drift(x,p,0.5*dt)
        x,p = pot.kick(x,p,dt)
        x,p = pot.drift(x,p,0.5*dt)
        ans.append([ x[0],x[1], p[0], p[1], pot.HJ(x,p) ])
    return np.array(ans)

if __name__ == "__main__": ## An example of how to use
```

```
pot = RotLogPot()
x = np.array([0.0, 0.9])
p = np.array([-1.0, 0.0 ])
xx = stepit(pot,x,p)
plt.plot(xx[:,0],xx[:,1])
```