M.Phys Option in Theoretical Physics: C6. Revision Problem Sheet, Quantum Field Theory

1.) (Field theories with global SO(n) symmetry)
a) The group SO(n) consists of the real $n\times n$ matrices R satisfying $R^T R = \mathbf{1}$ and $\det(R) = 1$. Consider infinitesimal transformations of the form $R = \mathbf{1} + iT$ and determine the allowed matrices T, that is, the Lie algebra of SO(n). Show that the matrices T_{ab}, labelled by an anti-symmetric pair of indices a, b and defined as ${(T_{ab})_c}^d = i(\delta_{ac}\delta_b^d - \delta_a^d\delta_{bc})$ (where $a, b, c, \ldots = 1, \ldots, n$) provide a basis for the Lie algebra. What is the dimension of this algebra?
b) A set of n scalar fields $\phi_a(x)$, where $a = 1, \ldots, n$, has a Lagrangian density $\mathcal{L} = \frac{1}{2}\sum_{a=1}^n \partial_\mu\phi_a\partial^\mu\phi_a - V$ with scalar potential $V = \frac{1}{2}m^2\sum_{a=1}^n \phi_a^2 + \frac{\lambda}{4}(\sum_{a=1}^n \phi_a^2)^2$. Show that this Lagrangian density is invariant under SO(n).
c) Compute the Noether currents of this SO(n) symmetry.
d) For $m^2 < 0$ and $\lambda > 0$ find the minima of the scalar potential and show that the SO(n) symmetry is broken to SO($n-1$). What is the number of Goldstone modes?

2.) (Theory with two complex scalars and local U(1) symmetry)
Consider a theory with two complex scalar fields ϕ_1 and ϕ_2 with masses m_1 and m_2 . The scalar fields have charges $q_1 = 1$ and $q_2 = 2$ under a local U(1) symmetry with associated vector field A_μ.
a) Write down the locally U(1) invariant Lagrangian density for these fields up to order four in the fields.
b) Write down the local U(1) symmetry transformations for all fields and explicitly show that the Lagrangian in a) is invariant.
c) Consider the limit where all cross-coupling terms between ϕ_1 and ϕ_2 vanish, that is, neglect all terms in the scalar potential which involve both ϕ_1 and ϕ_2. There are four different choices for the signs of m_1^2 and m_2^2. For each choice, find the minima of the scalar potential. In which of these cases is the local U(1) symmetry spontaneously broken?
d) For the broken cases, re-write the Lagrangian in terms of a massive vector field and the remaining physical scalars. What is the mass of the vector field and the scalars in each case? Comment on your results.

3.) (Canonical quantization of a scalar field)
Consider a free real scalar field ϕ with mass m.
a) Write down the general solution for this scalar field in terms of creation and annihilation operators and work out the commutators of creation and annihilation operators from the canonical commutation relations.
b) Write the (normal-ordered) Hamiltonian H in terms of creation and annihilation operators. Use this expansion and the results from a) to show explicitly that the time evolution of ϕ is governed by the equation $\dot{\phi}(x) = i[H, \phi(x)]$.
c) Show that $e^{-itH}a^\dagger(k)e^{itH} = e^{-ik_0 t}a^\dagger(k)$.
d) For a function $f(k)$ a *coherent state* $|f\rangle$ is defined by $|f\rangle = \exp\left(\int d^3\tilde{k}\, f(k)a^\dagger(k)\right)|0\rangle$. Show that $a(k)|f\rangle = f(k)|f\rangle$ and $e^{-iHt}|f\rangle = |e^{-ik_0 t}f\rangle$ for such a coherent state.

4.) (Wick's theorem)

a) For a real, free scalar field ϕ explicitly prove Wick's theorem for a product of four field, assuming its validity for a product of three fields.
b) Explain qualitatively the role of Wick's theorem in deriving Feynman rules for a quantum field theory.

5.) (Interacting scalar field theory)
Two real scalar fields ϕ_1 and ϕ_2 with masses $m_1^2 > 0$ and $m_2^2 > 0$ transform as $\phi_1 \to -\phi_1$ and $\phi_2 \to \phi_2$ under a $\mathbb{Z}_2$ symmetry.
a) Write down the most general $\mathbb{Z}_2$ invariant Lagrangian (with standard kinetic terms and up to quartic terms in the fields) for ϕ_1 and ϕ_2.
b) Derive the Feynman rules for the vertices in this theory by computing the appropriate Green functions.
c) Based on the results in b) and assuming negligible masses m_1 and m_2, find the amplitudes and total cross sections for $\phi_1\phi_1 \to \phi_1\phi_1$ scattering and $\phi_1\phi_1 \to \phi_2\phi_2$ scattering.