M.Phys Option in Theoretical Physics: C6. Revision Problem Sheet 1

Qu 1. A particle undergoes Brownian motion in one dimension. Its speed $v(t)$ as a function of time t satisfies the Langevin equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma v(t) + \eta(t)\,,$$

where $\eta(t)$ is a Gaussian random variable, characterised by the averages $\langle \eta(t)\rangle = 0$ and $\langle \eta(t_1)\eta(t_2)\rangle = \Gamma\delta(t_1 - t_2)$. Discuss the physical origin of each term in this equation. Show that, with initial condition $v(0) = 0$, the function

$$v_0(t) \equiv \int_0^t \mathrm{e}^{-\gamma(t-t')}\eta(t')\mathrm{d}t'$$

is a solution to the Langevin equation for $t > 0$.

Evaluate $\langle v_0(t_0)v_0(t_0 + t)\rangle$ for $t_0 \to \infty$. Consider both cases $t < 0$ and $t > 0$. Explain how the result enables one to express Γ in terms of temperature and other parameters characterising the system.

In the presence of an additional force on the particle, the Langevin equation has the modified form

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = f - \gamma v(t) + \eta(t)\,.$$

Show for constant f that the solution to this modified equation may be written in the form $v(t) = v_0(t) + u(t)$, and find $u(t)$.

A force is applied to the particle that depends linearly on its position $x(t)$, so $f = -\kappa x(t)$. For $\gamma^2 \gg \kappa$, the motion of the particle may be described approximately by the equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -gx(t) + n(t)\,,$$

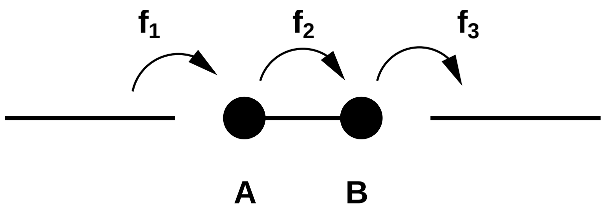
where $n(t)$ is a Gaussian random variable, characterised by the averages $\langle n(t)\rangle = 0$ and $\langle n(t_1)n(t_2)\rangle = G\delta(t_1 - t_2)$. Discuss the justification for this approximation and derive expressions for g and G in terms of γ, Γ and κ.

Qu 2. The time evolution of a stochastic system is represented by a master equation of the form

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \sum_m W_{nm}p_m(t)\,.$$

Explain briefly the meaning of this equation and discuss the assumptions on which it is based. What general conditions should the matrix elements W_{nm} satisfy?

A molecule lies between two atomic-scale contacts and conducts charge between them. A simple model of this situation is illustrated below. The model has three states: the molecule may be uncharged, or may carry a single charge at either site A or site B but not both. Charges hop between these sites, and between the sites and the contacts, at the rates indicated in the figure. (For example, a charge at site A has probability f_2 per unit time of hopping to site B.)



Write down a master equation for this model. For the system in equilibrium, calculate the occupation probabilities of the three states, and show that the average number of charges flowing through the molecule per unit time is

$$\frac{f_1f_2f_3}{f_1f_2 + f_1f_3 + f_2f_3}\,.$$

Consider the case $f_1 = f_2 = f_3 \equiv f$. The molecule is uncharged at time $t = 0$. Show that the probability $p(t)$ for it to be uncharged at a later time t is

$$p(t) = \frac{1}{3} + \frac{2}{3}\exp\left(-\frac{3}{2}ft\right)\cos\left(\frac{\sqrt{3}}{2}ft\right)\,.$$

Qu 3. Explain in outline how transfer matrices may be used in statistical mechanics to calculate properties of one-dimensional models with short-range interactions.

The Hamiltonian for a one-dimensional ferromagnetic Ising model is

$$H = -J \sum_{n=0}^{L-1} \sigma_n \sigma_{n+1} ,$$

where $J > 0$ and $\sigma_n = \pm 1$. Write down the transfer matrix for this model and find its eigenvalues and eigenvectors.

Let $F(\sigma_0, \sigma_L)$ be the free energy of this model, calculated for fixed values of σ_0 and σ_L, and let ΔF be the free energy difference given by

$$\Delta F = F(+1, -1) - F(+1, +1).$$

Calculate ΔF and put your result into the scaling form

$$\Delta F = k_{\mathrm{B}} T f(L/\xi),$$

giving an expression for the scaling function $f(x)$. Show that

$$\xi = \{\ln[\coth(\beta J)]\}^{-1} .$$

Find the leading temperature dependence of ΔF in the high and low temperature limits. Describe the ground states of H for each choice of boundary conditions, and hence interpret the behaviour of ΔF at low temperature.

Qu 4. Bosons move in a one-dimensional system of length L with periodic boundary conditions. Plane-wave basis states have the form

$$\varphi_k(x) = \frac{1}{\sqrt{L}} \mathrm{e}^{\mathrm{i}kx}$$

with $k = 2\pi n/L$, n integer. The operator $a_k^\dagger$ creates a boson in the state $\varphi_k(x)$.

Write down an expression for $a^\dagger(x)$, the creation operator for a boson at the point x, in terms of $a_k^\dagger$. Give also the inverse expression, for $a_k^\dagger$ in terms of $a^\dagger(x)$.

A two-boson state is defined by

$$|\Psi\rangle = \frac{1}{\sqrt{L}} \sum_k f_k a_k^\dagger \, a_{-k}^\dagger \, |0\rangle,$$

where $|0\rangle$ is the vacuum, with f_k real and $f_k = f_{-k}$.

Find the normalisation condition satisfied by f_k in order that $\langle \Psi | \Psi \rangle = 1$. Write the state $a_k^\dagger \, a_{-k}^\dagger |0\rangle$ as a two-particle wavefunction in coordinate space. Hence show that $|\Psi\rangle$ may be written in coordinate space as

$$\Psi(x_1, x_2) = \frac{\sqrt{2}}{L^{3/2}} \sum_k f_k \mathrm{e}^{\mathrm{i}k(x_1 - x_2)} .$$

The kinetic energy for the boson system is, with m the particle mass,

$$H_0 = \sum_k \frac{\hbar^2 k^2}{2m} a_k^\dagger \, a_k$$

and the interaction energy is

$$H_{\mathrm{I}} = \frac{1}{2} \sum_{kpq} V_q a_{k+q}^\dagger \, a_p^\dagger \, a_{p+q} \, a_k .$$

Find $\langle \Psi | H_0 | \Psi \rangle$ in terms of f_k, and find $\langle \Psi | H_{\mathrm{I}} | \Psi \rangle$ in terms of f_k and V_q.

The Hamiltonian for the system is $H = H_0 + H_{\mathrm{I}}$. Show that minimisation of

$$E = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

with respect to f_k yields a Schrödinger equation satisfied by f_k, of the form

$$\frac{\hbar^2 k^2}{m} f_k + \frac{1}{2} \sum_Q V_Q (f_{k-Q} + f_{k+Q}) = E f_k.$$

Qu 5. The spectrum of a Heisenberg ferromagnet at temperatures much less than the ordering temperature can be calculated using the following approximation. The spin operators at a given site are replaced by

$$\begin{aligned} S^x + iS^y \equiv S^+ &\simeq \sqrt{2S}\, a \\ S^x - iS^y \equiv S^- &\simeq \sqrt{2S}\, a^\dagger \\ S^z &\simeq S - a^\dagger a \end{aligned}$$

where the a, $a^\dagger$ are boson operators satisfying the commutation relations

$$[a, a^\dagger] = 1, \qquad [a, a] = 0, \qquad [a^\dagger, a^\dagger] = 0.$$

(a) Show that this replacement respects the spin commutation relations if $\langle a^\dagger a \rangle \ll S$.

(b) Substituting the above expressions into the Heisenberg Hamiltonian

$$H = -J \sum_{<ij>} \mathbf{S}_i \cdot \mathbf{S}_j,$$

where the summation is over neighbouring pairs of sites, obtain the Hamiltonian in terms of boson operators to order $a_i^\dagger a_j$.

For a simple cubic lattice show that the spectrum of excitations is

$$E(\mathbf{k}) = SJ \sum_\mu \{1 - \cos(\mathbf{k} \cdot \mathbf{e}_\mu)\}$$

where the $\mathbf{e}_\mu$ are the lattice vectors between nearest neighbours and $\mathbf{k}$ is the wave-vector of the excitation.

Show that $E \simeq \alpha k^2$ for small k. Evaluate α.

(c) Obtain an expression for the deviation of $\langle S^z \rangle$ from its maximum possible value S and show that it is indeed small at low temperatures for spatial dimension three or more. What happens in two dimensions?

$$\left[\text{Spin operators satisfy the commutation relations } [S^+, S^-] = 2S^z \text{ and } [S^z, S^\pm] = \pm S^\pm \right]$$

Past C6 exam questions that may be useful for revision [not for discussion in class]

Stochastic Processes: 2005, Qu. 5; 2002, Qu. 1; 1999, Qu. 2.

Many-Body Quantum Mechanics: 2007, Qu.8; 2006, Qu 5; 2006, Qu. 6; 2005, Qu. 7; 2005, Qu. 8.