M.Phys Option in Theoretical Physics: C6. Problem Sheet 5

1.) (Some useful kinematics)
a) Consider the decay of a particle with mass M into two particles with mass m. Use the general formula for the decay rate Γ in terms of the matrix element $\mathcal{M}$ derived in the lecture to show that

$$\Gamma = \frac{1}{16\pi M}\sqrt{1-\left(\frac{2m}{M}\right)^2}\,|\mathcal{M}|^2\ .$$

b) Two particle species A_1 and A_2 with masses m_1 and m_2 scatter in a process of the type $A_1 + A_2 \rightarrow A_1 + A_2$. For incoming momenta k_1, k_2 and outgoing momenta q_1, q_2 define the *Mandelstam variables* s, t, u by $s = (k_1 + k_2)^2$, $t = (k_1 - q_1)^2$ and $u = (k_1 - q_2)^2$. Show that $s + t + u = 2m_1^2 + 2m_2^2$.
c) Set $m = m_1 = m_2$ for simplicity. In the centre of mass frame, express the Mandelstam variables in terms of the total centre of mass energy E and the scattering angle θ (and the mass m).
d) Show that the matrix element $\mathcal{M}$ for the scattering process in b) can always be written as a function of s and t.

2.) A model with two real scalar fields ϕ_1 and ϕ_2 is described by the Lagrangian density $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 - M^2\phi_1^2 - m^2\phi_2^2 - \mu\phi_1\phi_2^2\right)$.
a) Derive the Feynman rule for the triplet vertex in this theory by computing the appropriate amputated Green function.
b) Assume that $M > 2m$ and compute the rate Γ for the decay of a ϕ_1 particle into two ϕ_2 particles at leading order.
c) Compute the matrix element for scattering of two ϕ_2 particles into two ϕ_2 particles at leading order and express it in terms of Mandelstam variables.
d) Write the matrix element in c) in terms of the total centre of mass energy E and the scattering angle θ. Compute the differential cross section $d\sigma/d\Omega$ for the process in c), assuming, for simplicity, that $E \gg m$.

3) For scalar electrodynamics
a) write down the Feynman diagrams relevant for the scattering of two positively charged scalars into two positively charged scalars through photon exchange at leading order in the coupling.
b) compute the matrix element of the process in a) and express it in terms of Mandelstam variables.
c) work out the differential cross section for the process in a) in terms of the total centre of mass energy E and the scattering angle θ.

4) Consider a real scalar ϕ with Lagrangian density $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \frac{\lambda}{4!}\phi^4$.
a) Compute the generating functional $W[J]$ perturbatively, neglecting terms of order λ^2 and higher.
b) Compute the generating functional $Z[J]$ for connected Green functions perturbatively to order λ.
c) Use the result from b) to determine the classical field ϕ_c in terms of the current J to order λ and invert the relation perturbatively to obtain J as a function of ϕ_c.
d) Compute the effective action to order λ. Interpret your result.

5) (Low-energy limit)

a) For the scattering process in 2c) determine an approximate form of the matrix element in the limit $M \gg E$.

b) In a field theory for ϕ_2 with the only interaction $\frac{\lambda_2}{4!}\phi_2^4$, which value of λ_2 is needed to produce the same approximate matrix element as in a)? Interpret your result.

6) For a free complex scalar field ϕ with mass m

a) compute the generating functional $W[J]$.

b) compute the two-point Green function from the result in a).