M.Phys Option in Theoretical Physics: C6. Problem Sheet 4

1.) Consider three real, free scalar fields ϕ_a with the same mass m and Lagrangian density $\mathcal{L} = \frac{1}{2}\sum_{a=1}^{3}\left(\partial_\mu\phi_a\partial^\mu\phi_a - m^2\phi_a^2\right)$.
a) Show that this theory has an SO(3) symmetry and derive the Noether currents and charges associated to this symmetry.
b) Quantise this theory by generalising the results for a single real scalar field from the lecture, that is, find the conjugate momenta, write down canonical commutation relations, expand fields in terms of creation and annihilation operators, work out their commutation relations and use the creation operators to construct the Fock space.
c) Find expressions for the conserved Noether charges in terms of creation and annihilation operators and use these expressions to verify that the Noether charges form an SO(3) algebra.
d) Compute the action of the Noether charges on one-particle states and show that the one-particle states with a given four-momentum k form a representation of SO(3). What is the "spin" of this representation?

2.) A free vector field A_μ with field strength $F_{\mu\nu}$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2$ (taken in Feynman gauge, $\lambda = 1$) can be expanded as

$$A_\mu(x) = \sum_{\alpha=0}^{3}\int d^3\tilde{k}\,\epsilon_\mu^{(\alpha)}(k)\left(a^{(\alpha)}(k)e^{-ikx} + a^{(\alpha)\dagger}(k)e^{ikx}\right) ,$$

where the polarisation vectors $\epsilon_\mu^{(\alpha)}(k)$ are as defined in the lecture.
a) Show that the standard canonical commutation relations for A_μ and its associated conjugate momenta π^ν follow, provided the creation and annihilation operators satisfy

$$[a^{(\alpha)}(k), a^{(\alpha')}(q)] = [a^{(\alpha)\dagger}(k), a^{(\alpha')\dagger}(q)] = 0 , \quad [a^{(\alpha)}(k), a^{(\alpha')\dagger}(q)] = -\eta^{\alpha\alpha'}(2\pi)^3 2w_{\mathbf{k}}\delta^3(\mathbf{k}-\mathbf{q}) .$$

b) Work out the Hamiltonian (in Feynman gauge) and express it in terms of creation and annihilation operators. Check that your result matches the one given in the lecture.
c) For physical states with one transversal state excited, verify explicitly that the energy does not depend on any non-transversal excitations.

3) A Greens function

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\tilde{\Delta}(k) \tag{1}$$

of the Klein-Gordon equation is defined as a function satisfying $(\Box + m^2)\Delta(x) = -i\delta^4(x)$.
a) Show that the Fourier transform $\tilde{\Delta}$ of Δ given by $\tilde{\Delta}(k) = i/(k^2 - m^2)$.
b) Insert the result from a) into Eq. (1) and discuss the various paths in the complex k_0 plane in relation to the two poles at $k_0 = \pm\sqrt{m^2 + \mathbf{k}^2}$ which can be chosen to carry out the k_0 integral.
c) Perform the k_0 integral for the paths in b) and show that for one choice of path $\Delta(x) = 0$ whenever $x_0 > 0$ and for another choice $\Delta(x) = 0$ whenever $x_0 < 0$. Interpret the physical relevance of these two Greens functions, as solutions to the Klein-Gordon equation with a

delta-function source.

4) For a real, free scalar field ϕ with mass m, evaluate $\int d^4x \langle 0|T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x)^4\right)|0\rangle$, using Wick's theorem and draw the associated Feynman diagrams. Focus on the connected part, Fourier transform $x_1,\dots,x_4$ to $k_1,\dots,k_4$ and express the result in terms of momentum space Feynman propagators.

5) Consider a complex, free scalar field ϕ. Apply Wick's theorem to $T(\phi(x_1)\phi(x_2)^\dagger)$ and prove the so-obtained relation explicitly. Do the same for $T(\phi(x_1)\phi(x_2))$.

6) Let's be adventurous and consider a field theory in *two dimensions*, with coordinates $(\sigma_\mu) = (\tau, \sigma)$, where $\mu, \nu, \dots = 0, 1$ and a collection of real scalar fields $X^a = X^a(\tau, \sigma)$, where $a, b, \dots = 0, \dots, n-1$. The Lagrangian density is given by $\mathcal{L} = -\frac{1}{2} g_{ab} \partial_\mu X^a \partial^\mu X^b$ with associated action $S = \int d^2\sigma\, \mathcal{L}$. Two-dimensional indices $\mu, \nu, \dots$ are lowered and raised with the two-dimensional Minkowski metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1)$ and its inverse. We take the metric g_{ab} to be the Minkowski metric in n dimensions, that is, $(g_{ab}) = \mathrm{diag}(-1, 1, \dots, 1)$. Also, we assume that the fields X^a are periodic with period $l = 2\pi$ in the spatial coordinate σ, so that $X^a(\tau, \sigma) = X^a(\tau, \sigma + l)$.
a) Derive the equations of motion for X^a using the variational principle.
b) Introduce "light-cone" coordinates $\sigma^\pm = \tau \pm \sigma$ and show that the equations of motion can be written in the form

$$\frac{\partial^2 X^a}{\partial \sigma^+ \partial \sigma^-} = 0 \, .$$

c) Show that the most general solution to this equation can be written as $X^a(\tau, \sigma) = X^a_-(\sigma^-) + X^a_+(\sigma^+)$ with

$$X^a_\pm(\sigma^\pm) = \frac{1}{2} x^a + p^a \sigma^\pm + \sum_{n \neq 0} \alpha^a_{\pm n} e^{-in\sigma^\pm} \tag{2}$$

where x^a, p^a are real constants and $\alpha^a_{\pm n}$ are complex constants with $\alpha^a_{\pm -n} = (\alpha^a_{\pm n})^\star$.
d) Determine the conjugate momenta for X^a and quantise the theory by imposing canonical commutation relations. Find the commutation relations for the operators x^a, p^a and $\alpha^a_{\pm n}$ in the expansion (2) such that the canonical commutation relations are satisfied.
e) Show that $\alpha^a_{\pm n}$ with $n < 0$ ($n > 0$) can be interpreted as creation (annihilation) operators and construct the Fock space. Then verify that the states $\alpha^0_{\pm n}|0\rangle$ for $n < 0$ have negative norm. Speculate on what might have gone wrong.