M.Phys Option in Theoretical Physics: C6. Problem Sheet 3

1.) Consider the set $\mathrm{Sl}(2,\mathbb{C})$ of complex 2×2 matrices with determinant one.
a) Show that this set forms a group.
b) Compute the Lie-algebra of this group, show that the dimension of this algebra is 6 and write down a basis of generators using the Pauli matrices.
c) By making an appropriate choice for the generators and computing their commutation relations, show that the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ is a representation of the Lorentz group Lie algebra.
d) As explained in the lecture, representations of the Lorentz group Lie algebra are classified by two spins (j_+, j_-). Which pair of spins does the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ corresponds to and why?

2.) Consider the group SO(3) of three-dimensional rotations, that is real 3×3 matrices R, satisfying $R^T R = \mathbf{1}_3$ and $\det(R) = 1$.
a) Show that the matrices R can be written in the form

$$R_{ij} = \cos(\alpha)(\delta_{ij} - n_i n_j) + \sin(\alpha)\epsilon_{ikj} n_k + n_i n_j$$

by i) thinking about a rotation with rotation axis $\mathbf{n}$ (where $|\mathbf{n}| = 1$) and rotation angle α and ii) by verifying that the above matrices R satisfy the defining relations of SO(3).
b) Find the explicit form of infinitesimal rotations, that is rotations with small angles α.
c) Determine the Lie algebra of SO(3) and a set of generators from the result in b) and verify that your answer coincides with the result given in the lecture.
d) Show that the Lie-algebra of SO(4) (the group of real 4×4 matrices R satisfying $R^T R = \mathbf{1}_4$ and $\det(R) = 1$) consists of anti-symmetric 4×4 matrices. Choose a convenient basis of generators (for example, so that 3 of the generators corresponds to the three-dimensional rotations $\mathrm{SO}(3) \subset \mathrm{SO}(4)$).

3.) The Lagrangian density $\mathcal{L} = \mathcal{L}(\partial_\mu \phi_a(x), \phi_a(x))$ for a set of fields $\phi_a = \phi_a(x)$ is assumed to be invariant under the infinitesimal transformation

$$\phi_a \to \phi_a - i t^i (T_i)_a{}^b \phi_b \ ,$$

where T_i are the generators of a Lie algebra with commutators $[T_i, T_j] = i f_{ij}{}^k T_k$ and t^i are the symmetry parameters.
a) Find the conserved currents $j_{i\mu}$ and the associated conserved charges Q_i for this symmetry.
b) Now consider the quantized version of this theory, where $[\phi_a(t,\mathbf{x}), \phi_b(t,\mathbf{y})] = [\pi^a(t,\mathbf{x}), \pi^b(t,\mathbf{y})] = 0$ and $[\pi^a(t,\mathbf{x}), \phi_b(t,\mathbf{y})] = -i\delta^a_b \delta^3(\mathbf{x}-\mathbf{y})$. Compute the commutators $[Q_i, Q_j]$ for the operator versions of the conserved charges Q_i and comment on your result.

4.) A model with a real scalar field σ and three other real scalar fields $\boldsymbol{\phi} = (\phi_a)$ is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \phi_a \partial^\mu \phi_a\right) - V(\sigma, \boldsymbol{\phi}) \ , \quad V = \frac{m^2}{2}(\sigma^2 + \boldsymbol{\phi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\phi}^2)^2 \ .$$

a) Show that this Lagrangian is unvariant under SO(4) acting on the four-dimensional vectors $(\sigma, \boldsymbol{\phi})$.

b) Using the result from 2 d) or otherwise, show that infinitesimal SO(4) transformations of the fields can be written as

$$\sigma \to \sigma + \boldsymbol{\beta} \cdot \boldsymbol{\phi} \,, \quad \boldsymbol{\phi} \to \boldsymbol{\phi} + \boldsymbol{\alpha} \times \boldsymbol{\phi} - \boldsymbol{\beta}\sigma \,,$$

for suitable defined small symmetry parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
c) Find the six conserved currents and charges.
d) Analyze spontaneous breaking of the SO(4) symmetry in the case where $m^2 < 0$. In particular, find the vacua, determine the unbroken sub-group and the Goldstone modes. (Hint: For the two final tasks, choose a minimum of the potential for which $\boldsymbol{\phi} = 0$ and $\sigma \neq 0$.)

5.) a) Derive the energy-momentum tensor $T^{\mu\nu}$ from the Lagrangian formulation of the free Maxwell theory, using the general procedure explained in the lecture.
b) Given that $T^{\mu\nu}$ is conserved show that

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$$

is still conserved provided the tensor $K^{\rho\mu\nu}$ is anti-symmetric in its first two indices.
c) Choosing $K^{\rho\mu\nu} = F^{\mu\rho} A^\nu$ show that $\tilde{T}^{\mu\nu}$ is a symmetric tensor.
d) Write down the conserved charges associated to $\tilde{T}^{\mu\nu}$ and (by writing them in terms of $\mathbf{E}$ and $\mathbf{B}$) show that they yield the standard expressions of the electromagnetic energy and momentum densities.

6.) Write down the conserved four-momentum P^μ and angular momentum $M^{\mu\nu}$ for a free, real scalar field ϕ with mass m.
a) Using the canonical commutation relations, show that

$$[P^\mu, \phi(x)] = -i\frac{\partial \phi(x)}{\partial x_\mu} \,, \quad [M^{\mu\nu}, \phi(x)] = -i\left(x^\mu \frac{\phi(x)}{\partial x_\nu} - x^\nu \frac{\partial \phi(x)}{\partial x_\mu}\right) \,.$$

b) Determine the commutators $[M^{\mu\nu}, M^{\rho\sigma}]$, $[M^{\mu\nu}, P^\rho]$ and $[P^\mu, P^\nu]$ and show that the set of operators $\{P^\mu, M^{\mu\nu}\}$ forms a closed algebra under the commutator bracket.

7.) Consider the Lagrangian for a single, free complex scalar field with mass m and charge q under a U(1) symmetry.
a) Derive the conserved U(1) current and charge.
b) Write the conserved charge in terms of creation and annihilation operators.
c) Use the result from b) to explicitly work out the charge of single and multiple particle states.