M.Phys Option in Theoretical Physics: C6. Problem Sheet 2

Qu 1. What is: (a) a Markov process; (b) a Markov chain?

A salesman's territory consists of three cities A, B and C. Each day is spent selling in one of the cities. He never sells in the same city on successive days. If he sells in city A, then on the next day he sells in city B. However, if he sells in either B or C, then on the next day he is twice as likely to sell in city A as in the other city.

Write down the transition matrix for this process. In the long run, how often does he sell in each of the cities? Suppose that the salesman happens to sell in city B on a particular day. What is the probability that he is again selling in B: (i) two days later; (ii) n days later?

Qu 2. A lattice model for non-ideal gas is defined as follows. The sites i of a lattice may be empty or occupied by at most one atom, and the variable n_i takes the values $n_i = 0$ and $n_i = 1$ in the two cases. There is an attractive interaction energy J between atoms that occupy neighbouring sites, and a chemical potential μ. The model Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle ij \rangle} n_i n_j - \mu \sum_i n_i \,, \tag{1}$$

where $\sum_{\langle ij \rangle}$ is a sum over neighbouring pairs of sites.

Describe briefly how the *transfer matrix method* may be used to calculate the statistical-mechanical properties of one-dimensional lattice models with short range interactions. Illustrate your answer by explaining how the partition function for a one-dimensional version of the lattice gas, Eq. (1), defined on a lattice of N sites with periodic boundary conditions, may be evaluated using the matrix

$$\mathbf{T} = \begin{pmatrix} 1 & \mathrm{e}^{\beta\mu/2} \\ \mathrm{e}^{\beta\mu/2} & \mathrm{e}^{\beta(J+\mu)} \end{pmatrix} .$$

Derive an expression for $\langle n_i \rangle$ in the limit $N \to \infty$, in terms of elements of the eigenvectors of this matrix.

Show that

$$\langle n_i \rangle = \frac{1}{1 + \mathrm{e}^{-2\theta}} \,,$$

where

$$\sinh(\theta) = \exp(\beta J/2) \sinh(\beta[J+\mu]/2) \,.$$

Sketch $\langle n_i \rangle$ as a function of μ for $\beta J \gg 1$, and comment on the physical significance of your result.

Qu 3. The Hamiltonian for a one-dimensional ferromagnetic Ising model is

$$H = -J \sum_{n=0}^{L-1} \sigma_n \sigma_{n+1} \,,$$

where $J > 0$ and $\sigma_n = \pm 1$. Write down the transfer matrix for this model and find its eigenvalues and eigenvectors.

Let $F(\sigma_0, \sigma_L)$ be the free energy of this model, calculated for fixed values of σ_0 and σ_L, and let ΔF be the free energy difference given by

$$\Delta F = F(+1, -1) - F(+1, +1)$$

Calculate ΔF and put your result into the scaling form

$$\Delta F = k_{\mathrm{B}} T f(L/\xi),$$

giving an expression for the scaling function $f(x)$. Show that

$$\xi = \{\ln[\coth(\beta J)]\}^{-1}$$

Find the leading temperature dependence of ΔF in the high and low temperature limits. Describe the ground states of H for each choice of boundary conditions, and hence interpret the behaviour of ΔF at low temperature.

Qu 4. The one-dimensional, nearest-neighbour, spin-1 Ising model has the Hamiltonian

$$H = -J \sum_{n=1}^{N} S_n S_{n+1}$$

where the spin S_n at site n takes the values $S_n = -1, 0, 1$, the total number of sites is N and periodic boundary conditions are imposed so that $S_1 \equiv S_{N+1}$. Use the transfer matrix method to show that the free energy per site f of this model in the thermodynamic limit is given by

$$f = -k_{\mathbf{B}} T \ln \left\{ \cosh(\beta J) + \frac{1}{2} + \left[\left(\cosh(\beta J) - \frac{1}{2} \right)^2 + 2 \right]^{1/2} \right\}$$

Find the behaviour of f in the high and low temperature limits. Give a physical interpretation of your results.