M.Phys Option in Theoretical Physics: C6. Problem Sheet 1

Qu 1. Consider paths $\mathbf{X} = \mathbf{X}(\tau)$, where τ is a parameter, and the functional

$$l[\mathbf{X}] = \int_{\tau_0}^{\tau_1} d\tau\, n(\mathbf{X})\sqrt{\frac{d\mathbf{X}}{d\tau}\cdot\frac{d\mathbf{X}}{d\tau}}\ ,$$

where $n = n(\mathbf{X})$ is a function. (The minima of this functional can be interpreted as light rays propagating in a medium with refractive index n.)
a) Vary the above functional and derive the differential equation which has to be satisfied by minimal paths $\mathbf{X}$.
b) Consider a two-dimensional situation with paths $\mathbf{X}(\tau) = (X(\tau), Y(\tau))$ in the x, y plane and a function $n = n_0 + (n_1 - n_0)\,\theta(x)$. (The Heaviside function $\theta(x)$ is defined to be 0 for $x < 0$ and 1 for $x \geq 0$. Recall that $\theta'(x) = \delta(x)$.) Solve the differential equation in a) for this situation, using the coordinate x as parameter τ along the path.
c) Show that the solution in b) leads to the standard law for refraction at the boundary between two media with refractive indices n_0 and n_1.

Qu 2. The gamma function Γ is defined by

$$\Gamma(s) = \int_0^\infty dx\, x^{s-1} e^{-x}\ .$$

a) Show that $\Gamma(1) = 1$ and $\Gamma(s+1) = s\Gamma(s)$. (Hence $\Gamma(n+1) = n!$)
b) Through a suitable change of the integration variable, rewrite the above expression for the gamma function in the form $\Gamma(s) = f(s)\int_0^\infty dy\ \exp(-A(y)/\xi(s))$ and indentify the functions $f(s)$, $A(y)$ and $\xi(s)$.
c) Evaluate $\Gamma(s)$ in the steepest descent approximation.

Qu 3. Consider the generating function

$$W_\lambda[\mathbf{A}, \mathbf{J}] = \int d^n\mathbf{y}\ \exp\left(-\frac{1}{2}\mathbf{y}^T\mathbf{A}\mathbf{y} - \lambda V(\mathbf{y}) + \mathbf{J}\mathbf{y}\right)$$

with a quartic interaction $V(\mathbf{y}) = \frac{1}{4!}\sum_{i=1}^n y_i^4$.
a) Using Wick's theorem, compute the two-point Green's function $\mathcal{G}_{ij}^{(2)}$ to order λ^2, neglecting all terms of order λ^3 or higher.
b) Draw the Feynman diagrams associated to the result in a).
c) Compute the two-point function $\langle y_i y_j\rangle_\lambda$ to order λ^2 and show that it only consists of connected Feynman diagrams.

Qu 4. In this question the objective is to evaluate the Feynman path integral in one of the relatively few cases, besides those treated in lectures, for which exact results can be obtained. The system we consider consists of a particle of mass m moving on a circle of circumference L. The quantum Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

and wavefunctions obey $\psi(x+L) = \psi(x)$. You are asked to evaluate a matrix element of the Boltzmann factor

$$\langle x_1|\exp(-\beta H)|x_2\rangle\ .$$

First, do this in a standard way, by finding the eigenfunctions and eigenvalues of H, and show that

$$\langle x_1|\exp(-\beta H)|x_2\rangle = \sum_{n=-\infty}^{\infty}\frac{1}{L}\exp\left(-\frac{\beta(2\pi n)^2\hbar^2}{2mL^2} + 2\pi\mathrm{i}n\frac{[x_1 - x_2]}{L}\right)\ . \tag{1}$$

Next, approach this using a path integral in which paths $x(\tau)$ for $0 \leq \tau \leq \beta\hbar$ satisfy the boundary conditions $x(0) = x_1$ and $x(\beta\hbar) = x_2$. The special feature of a particle moving on a circle is that such paths may wind any integer number l times around the circle. To build in this feature, write

$$x(\tau) = x_1 + \frac{\tau}{\beta\hbar}[(x_2 - x_1) + lL] + s(\tau),$$

where the contribution $s(\tau)$ obeys the simpler boundary conditions $s(0) = s(\beta\hbar) = 0$ and does *not* wrap around the circle. Show that the Euclidean action for the system on such a path is

$$S[x(\tau)] = S_l + S[s(\tau)] \qquad \text{where} \qquad S_l = \frac{m}{2\beta\hbar}[(x_2 - x_1) + lL]^2 \qquad \text{and} \qquad S[s(\tau)] = \int_0^{\beta\hbar} \mathrm{d}\tau \frac{m}{2}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 .$$

Hence show that

$$\langle x_1 | \exp(-\beta H) | x_2 \rangle \ = \mathcal{Z}_0 \sum_{l=-\infty}^{\infty} \exp\left(-\frac{m}{2\beta\hbar^2}[(x_1 - x_2) + lL]^2\right) \tag{2}$$

where $\mathcal{Z}_0$ is the diagonal matrix element $\langle x|\mathrm{e}^{-\beta H}|x\rangle$ for a *free* particle (i.e. without periodic boundary conditions) moving in one dimension. By taking results from your lecture notes for the time evolution operator for a free particle, and making the substitution $\mathrm{i}t = \beta\hbar$, show that

$$\mathcal{Z}_0 = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{1/2} .$$

Finally, of course, you should show that the expressions in Eq. (1) and Eq. (2) are indeed equal. To do so, you should use the *Poisson summation formula*, which is

$$\sum_{l=-\infty}^{\infty} \delta(y - l) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} n y}$$

(think about how to justify this). Introduce the left hand side of this expression into Eq. (2) by using the relation, valid for any smooth function $f(y)$,

$$\sum_{l=-\infty}^{\infty} f(l) = \int_{-\infty}^{\infty} \mathrm{d}y \sum_{l=-\infty}^{\infty} \delta(y - l) f(y) \, ,$$

substitute the right hand side of the summation formula, carry out the (Gaussian) integral on y, and hence establish the required equality.

Qu 5. This question is concerned with the central limit theorem.
(i) Show explicitly that for $N \gg 1, pN \gg 1$ the binomial distribution

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}, \quad p + q = 1$$

becomes

$$P_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \ \exp\left(\frac{-(n- < n >)^2}{2\sigma^2}\right)$$

where $\sigma^2 = Npq$. Check that the same result follows from the central limit theorem.
(ii) Consider a random walk in one dimension, for which the probability of taking a step of length $x \to x + dx$ is

$$f(x)dx = \frac{1}{\pi} \frac{\gamma}{x^2 + \gamma^2} \, dx.$$

Find the probability distribution for the total displacement after N steps. Does it satisfy the central limit theorem? Should it? What are the cumulants of this distribution?

Qu 6. Let $y = \pm 1$. Show that

$$P_{1|1}(y, t \mid y', t') = \frac{1}{2}\left\{1 + e^{-2\gamma(t-t')}\right\}\delta_{y,y'} + \frac{1}{2}\left\{1 - e^{-2\gamma(t-t')}\right\}\delta_{y,-y'} \tag{3}$$

obeys the Chapman-Kolmogorov equation.

Show that

$$P_1(y, t) = \frac{1}{2}(\delta_{y,1} + \delta_{y,-1}) \tag{4}$$

is a stationary solution. Write $P_{1|1}$ as a 2×2 matrix and formulate the Chapman-Kolmogarov equation as a property of that matrix.

Qu 7. This question is about a continuous random walk, also known as a Wiener process.
Show that for $-\infty < y < \infty$ and $t_2 > t_1$ the Chapman-Kolmogarov equation is satisfied for

$$P_{1|1}(y_2, t_2 \mid y_1, t_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp - \left\{ \frac{(y_2 - y_1)^2}{2(t_2 - t_1)} \right\}. \tag{5}$$

Choose $P_1(y_1, 0) = \delta(y_1)$. Show that for $t > 0$

$$P_1(y, t) = \frac{1}{\sqrt{2\pi t}} \exp \left\{ \frac{-y^2}{2t} \right\}. \tag{6}$$

Show that $P_1(y, t)$ obeys the diffusion equation

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial y^2} \tag{7}$$

for $D = \frac{1}{2}$. What is the solution for arbitrary $D > 0$?

Qu 8. A particle suspended in a fluid undergoes Brownian motion in one dimension with position $x(t)$ and velocity $v(t)$. This motion is modelled by the Langevin equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + \eta(t),$$

where $\eta(t)$ is a Gaussian random variable characterised completely by the averages $\langle \eta(t) \rangle = 0$ and $\langle \eta(t_1)\eta(t_2) \rangle = \Gamma \delta(t_1 - t_2)$. Discuss the physical origin of each of the terms in the Langevin equation.

What is meant by the term *Markov process*? Illustrate your answer by discussing which of the following are Markov processes: (a) $v(t)$ alone; (b) $x(t)$ alone; (c) $v(t)$ and $x(t)$ together.

Show that for $t > 0$

$$x(t) = \frac{v(0)}{\gamma} (1 - \mathrm{e}^{-\gamma t}) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \; \mathrm{e}^{-\gamma(t_1 - t_2)} \, \eta(t_2)$$

is a solution of the Langevin equation with initial condition $x(0) = 0$. Calculate the average $\langle x(t)\, v(t) \rangle$ and discuss its limiting behaviour at short and long times.