Minimal models for chaotic quantum dynamics in spatially extended many-body systems

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Joint work with Amos Chan and Andrea De Luca

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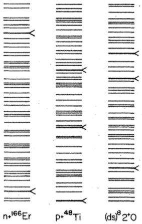
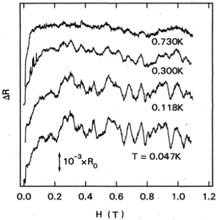
Studies of ‘generic’ quantum systems

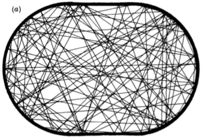
Nuclear physics



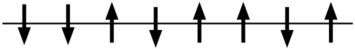
Mesoscopic conductors



Low-D systems



Spatially extended many-body systems



Characterising spectra

Evolution operator $W(t)$, eigenvalues $e^{i\theta_n t}$

Spectral form factor $K(t) = \left\langle \sum_{nm} e^{i(\theta_n - \theta_m)t} \right\rangle$

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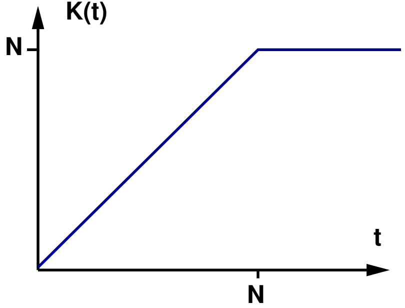
Unitary $N \times N$ random matrices



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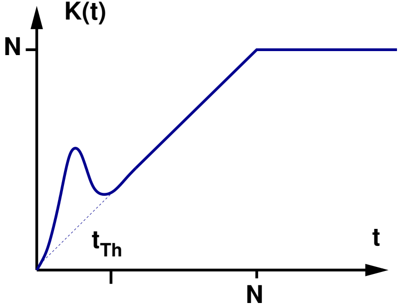
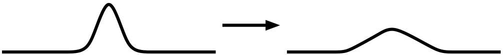
Mesoscopic conductor: Thouless time

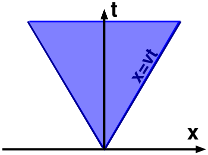


Characterising dynamics

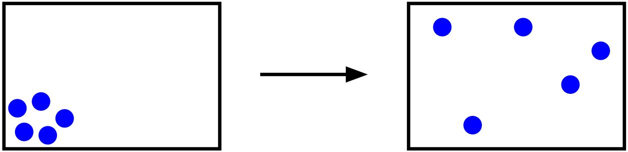
Hydrodynamics & conserved densities



Dynamics of quantum information



Equilibration under unitary dynamics



Speed limits without relativity

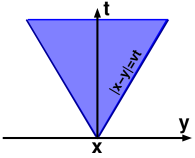
Lieb-Robinson bound (1972)

Max propagation speed v for disturbances in short-range lattice models

For local observables at x and y

$$[O(y,t), O(x)]$$

small outside lightcone



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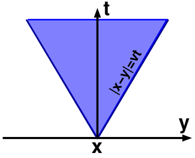
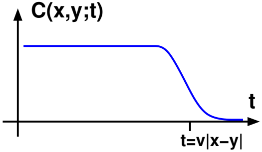


Out-of-time-order correlator (OTOC)

$$C(x,y;t) \equiv [O(y,t)O(x)O(y,t)O(x)]_{\mathrm{av}}$$

E.g. with $\mathrm{Tr}\, O(x) = 0$
and $O(x)^2 = \mathbb{1}$
& likewise for $O(y)$



Entanglement dynamics

Quantifying 'equilibration' under unitary dynamics

Density matrix $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ for full system

— pure state preserved under time evolution

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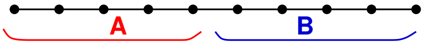
Reduced density matrix $\rho_A(t) = \mathrm{Tr}_B \rho(t)$



Entropy of sub-system may grow with time & saturate at long times

Aim: solvable models for ergodic phase

Simple physics: Eliminate conserved densities $\Rightarrow$ time-dept evolution operator

Simple solution: Random matrices & spatial structure

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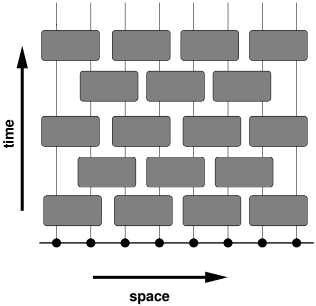
Inspiration: Random unitary circuits



Nahum, Ruhman, Vijay and Haah, PRX (2017)
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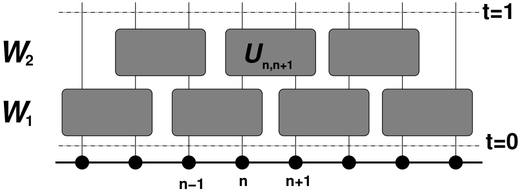
Minimal model

L-site lattice of q-state 'spins'

Floquet operator W is $q^L \times q^L$ unitary matrix



Each $q^2 \times q^2$ unitary $U_{n,n+1}$ independently Haar-distributed

Solve for $q \to \infty$

Behaviour of Floquet model

Is behaviour consistent with ergodic phase?

Relaxation of local observables

Dynamics of quantum information?

Out-of-time-order correlator

Entanglement growth

Spectral correlations?

'Thouless time' in many-body system

Relaxation of local observables

Local operator in q-state Hilbert space at site: $O(x)$

with $\mathrm{Tr}\, O(x) = 0$ and $O(x)^2 = \mathbb{1}_q$.

Define

$$O(x,t) = W(t) O(x) W^\dagger(t) \qquad [\ldots]_{\rm av} \equiv q^{-L} \mathrm{Tr} \ldots$$

Want $\quad [O(x,t) O(x)]_{\rm av}$

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Find for $q \to \infty$ $\quad [O(x,t)O(x)]_{\mathrm{av}} = \begin{cases} 1 & t = 0 \\ 0 & t > 0 \end{cases}$

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Short times and finite q: $\quad [O(x,t)O(x)]_{\mathrm{av}} = \begin{cases} 1 & t=0 \\ 0 & t=1 \\ q^{-7} & t=2 \\ 16q^{-11} & t=3 \end{cases}$

Out-of-time-order correlator

Find for $q \to \infty$

$$[O(y,t)O(x)O(y,t)O(x)]_{\mathrm{av}} = \begin{cases} 1 & |t| < |x-y|/2 \\ 0 & |t| \geq |x-y|/2 \end{cases}$$

Butterfly velocity $v = 2$

Entanglement growth in Floquet model

Initial state $|\psi\rangle$ – product state in site basis

Reduced density matrix $\rho_A(t) = \mathrm{Tr}_B W(t)|\psi\rangle\langle\psi|W^\dagger(t)$

Réyni entropies $e^{-(\alpha-1)S_\alpha(t)} = \mathrm{Tr}_A[\rho_A(t)^\alpha]$

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with $\alpha = 2$ or 3 and q large

Find $$\langle e^{-(\alpha-1)S_\alpha(t)}\rangle = \begin{cases} f_\alpha(t)q^{-2(\alpha-1)t} & t \leq L/4 \\ K_\alpha q^{-(\alpha-1)L/2} & t > L/4 \end{cases}$$

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Interpretation:

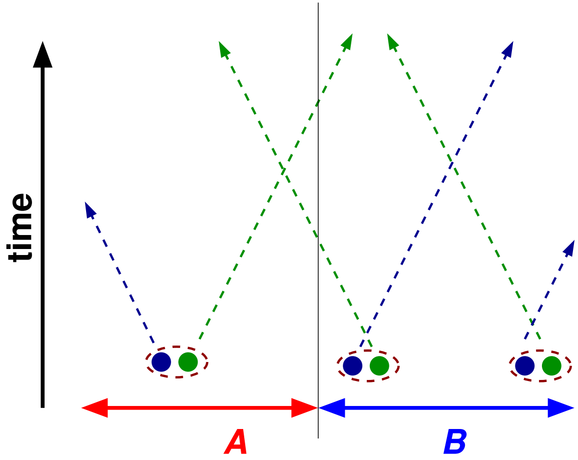
Entanglement at time t has range $2t$

$\Rightarrow$ $\rho_A(t)$ has q^{2t} non-zero eigenvalues, each $\mathcal{O}(q^{-2t})$

Entanglement spreads at speed $v = 2$

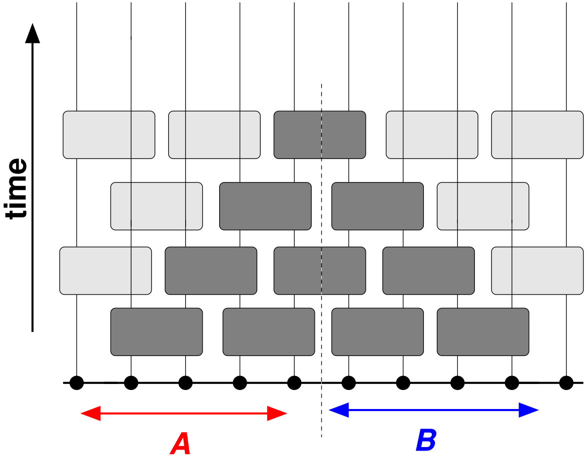
Entanglement growth after quench in integrable systems

— from quasiparticle dynamics



Cardy and Calabrese

Entanglement growth in quantum circuits



What is lost in $q \to \infty$ limit?

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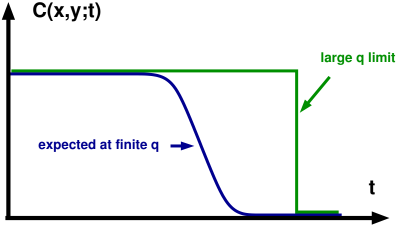
OTOC: Front is sharp, not diffuse



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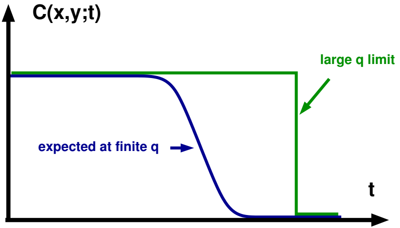

Velocities: 'Naive' value for all speeds
(butterfly, entanglement spreading . . .)

Spectral form factor

Evolution operator $W(t)$ with eigenvalues $\{e^{i\theta_n}\}$

Spectral form factor $K(t) = \sum_{m,n} e^{i(\theta_m - \theta_n)t}$

Large q $\Rightarrow$ random matrix behaviour in Floquet model

$K(t) = t$ for $0 < t \ll q^L$

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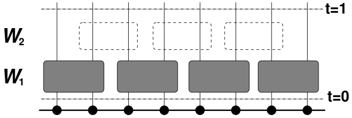
— consequence of coupling



Without W_2 find instead

$$K(t) = t^{L/2}$$

Origin of RMT behaviour?

Constructive interference of diagonal terms in double sum on paths
(diagonal approximation $\sim$ diffusons)

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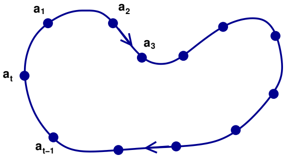
$$K(t) = \left\langle \sum_{nm} e^{i(\theta_n - \theta_m)t} \right\rangle = \left\langle \mathrm{Tr}[W(t)]\mathrm{Tr}[W^\dagger(t)] \right\rangle$$

$$\mathrm{Tr}[W(t)] \equiv \sum_{a_1 \ldots a_t} W_{a_1 a_2} W_{a_2 a_3} \ldots W_{a_t a_1}$$



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Constructive interference if path $b_1 b_2 \ldots b_t$ though Fock space
is reversed copy of path $a_1 a_2 \ldots a_t$

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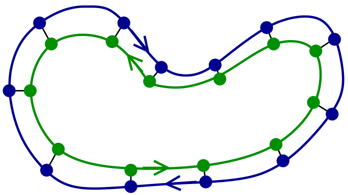
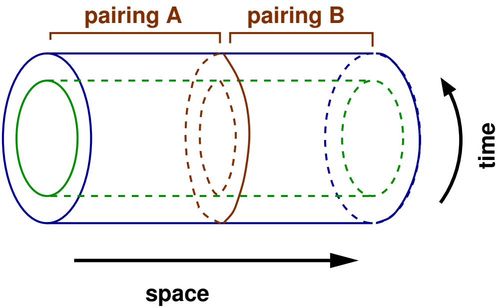
Pictorially:



t possible pairings

New possibilities in many-body systems?

Spatial domains with distinct pairings between time-reversed paths



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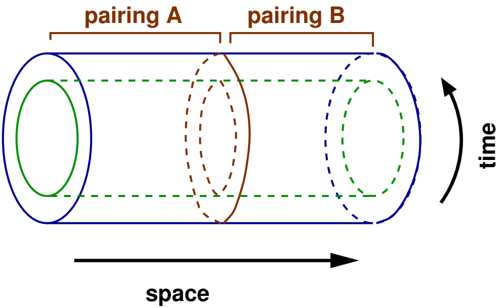


Equivalence to t-state Potts model:

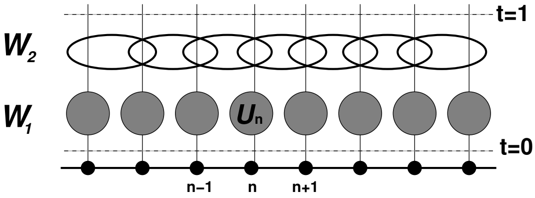
t pairings in each domain

& statistical cost for domain walls

Model of weakly-coupled chaotic grains

L-site lattice of q-state 'spins'

Floquet operator $W \equiv W_2 \cdot W_1$ is $q^L \times q^L$ unitary matrix



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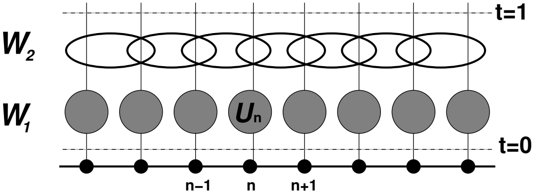

Each $q \times q$ unitary U_n independently Haar-distributed

$W_1 = U_1 \otimes U_2 \otimes \ldots \otimes U_L$

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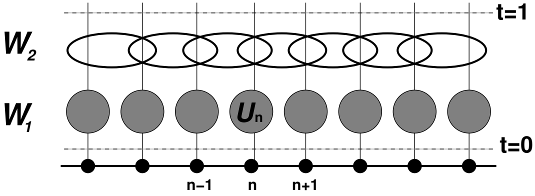

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W_2 diagonal in site basis $|a_1, a_2, \ldots a_L\rangle$ with phase $\sum_{n=1}^{L} \varphi_{a_n, a_{n+1}}$

$\{\varphi_{a_n, a_{n+1}}\}$ indept Gaussian random variables, zero mean

Coupling strength $\langle [\varphi_{a_n, a_{n+1}}]^2 \rangle \equiv \varepsilon$

Many-body 'Thouless time'

Small $t \quad \Rightarrow \quad L$ uncoupled sites $\quad \Rightarrow \quad K(t) = t^L$

Large $t \quad \Rightarrow \quad$ all sites coupled $\quad \Rightarrow \quad K(t) = t$

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Exact mapping to t-state Potts ferromagnet

Potts coupling $\beta J \equiv \varepsilon t \quad \Rightarrow \quad K(t) = Z_{\rm Potts}$

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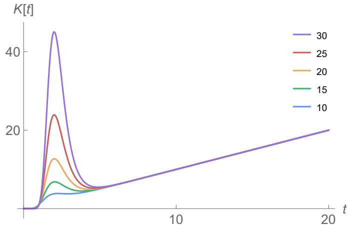

$K(t)$ vs t for $q \to \infty$

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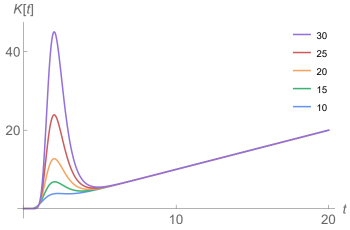

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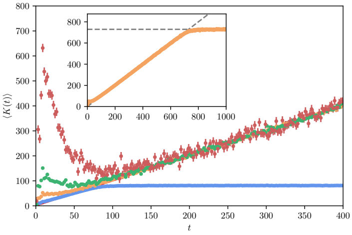

K(t) vs t for $q = 3$, $L = 4 - 10$

Summary

Floquet models at large q give solvable ergodic phase

Systematic calculations for $q \to \infty$

Rapid local relaxation

Light cone in OTOC

Ballistic growth of entanglement

Many-body Thouless time

Crossover between uncoupled sites and single RMT behaviour

Calculation of entanglement in Floquet model

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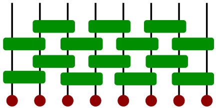
Consider $\mathrm{Tr}_A[\rho_A(t)]^2$ with $\rho_A = \mathrm{Tr}_B[W(t)|\psi\rangle\langle\psi|W^\dagger(t)]$



$W(t)|\psi\rangle$

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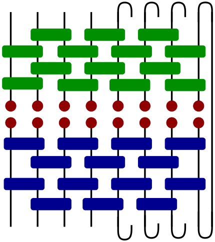

$\rho_A(t)$

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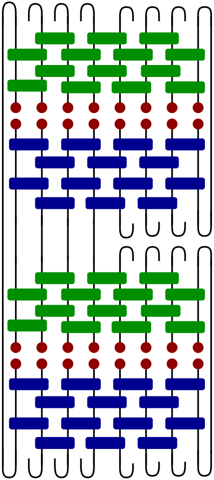

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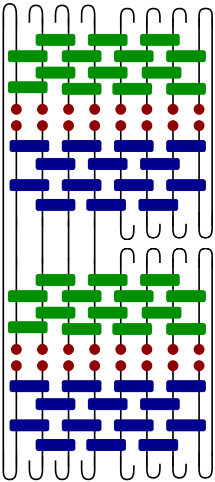
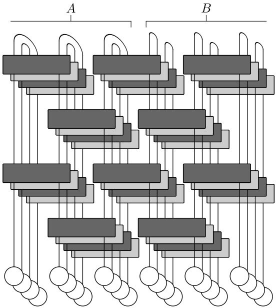


Alternatively draw as



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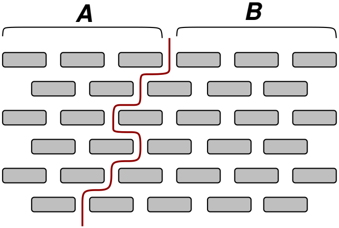
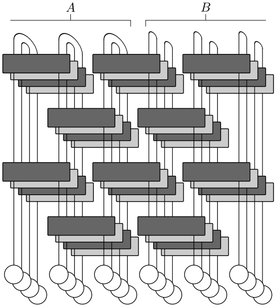
Represent contributions after averaging
via domains for different pairings





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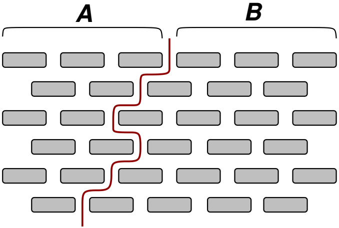


$$\mathrm{Tr}_A[\rho_A(t)]^2 = 4^t q^{-2t}$$

4^t no. of directed paths of length t

q^{-2} statistical cost of path per step

Summary

Floquet models at large q give solvable ergodic phase

Systematic calculations for $q \to \infty$

Rapid local relaxation

Light cone in OTOC

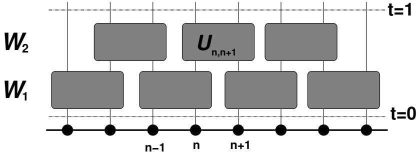
Ballistic growth of entanglement

Many-body Thouless time

Crossover between uncoupled sites and single RMT behaviour

Essentials of calculation: avges of unitary matrices

Floquet operator W is $q^L \times q^L$ unitary matrix



Each $q^2 \times q^2$ unitary $U_{n,n+1}$ independently Haar-distributed

Need

$$\langle [U]_{a_1,b_1} \ldots [U]_{a_t,b_t} \times [U^\dagger]_{\alpha_1,\beta_1} \ldots [U^\dagger]_{\alpha_t,\beta_t} \rangle = V_{P,P'} \prod_{j=1}^{t} \delta_{a_j,\alpha_{P(j)}} \delta_{b_j,\beta_{P'(j)}}$$

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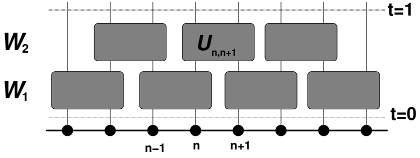

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Large q: $P = P'$ dominates

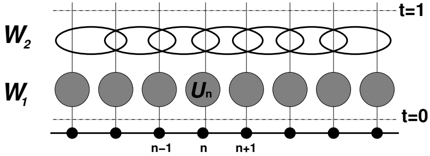
($[U]_{ab}$ almost independent Gaussian variables)

Diagrammatic technique for single-pcle problem: Brouwer+Beenakker (1996)

Calculating $K(t)$ in Floquet model at $q \to \infty$

$K(t) = \langle [\mathrm{Tr}\, W(t)] \cdot [\mathrm{Tr}\, W^\dagger(t)] \rangle$ notation: $W(t) \equiv [W]^t$

Avge on W_1



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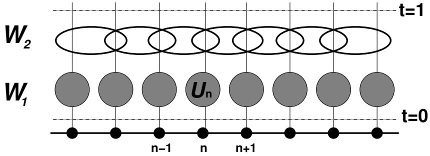

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t contributions of this form, for $s = 1 \ldots t$

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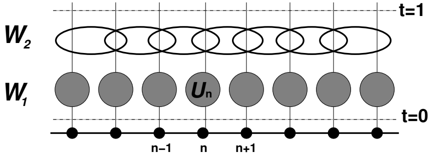

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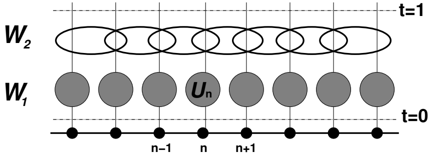

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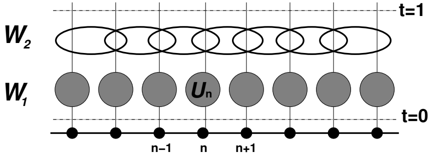

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t-state Potts model