Quantum Theory of Condensed Matter: Problem Set 1

Qu.1 Consider two quantum spins, $\mathbf{S}_1$ and $\mathbf{S}_2$, of magnitude S, interacting via the Hamiltonian

$$\mathcal{H} = -J\mathbf{S}_1.\mathbf{S}_2 - H(S_1^z + S_2^z)$$

with $J > 0$.

(i) Use the standard theory for addition of angular momenta to find the exact energy levels.

(ii) Use the Holstein-Primakoff transformation and harmonic approximation to calculate the low-lying excitation energies.

(iii) Compare the exact and approximate calculations.

Qu.2 Consider a Bose gas at zero temperature. If the bosons are non-interacting, all particles occupy the lowest energy single-particle state. Repulsive interactions cause a *depletion* of the condensate. Calculate the fraction of bosons not in the single-particle ground state, as follows.

The operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are boson destruction and creation operators for the single-particle state with wavevector $\mathbf{k}$. They satisfy $[a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger] = 1$. There are short-range interactions between bosons with strength parameterised by u. The Hamiltonian for N bosons moving in a d-dimensional box of volume L^d, with number density $n = N/L^d$, is to leading order

$$H = \frac{unN}{2} + \frac{1}{2}\sum_{\mathbf{k}\neq 0}{}' \left[(\frac{\hbar^2 k^2}{2m} + un)(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}) + un(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} a_{-\mathbf{k}}) \right]$$

(i) Use a Bogoluibov transformation of the form

$$a_{\mathbf{k}} = \cosh(\theta_k)\alpha_{\mathbf{k}} + \sinh(\theta_k)\alpha_{-\mathbf{k}}^\dagger$$

to write the Hamiltonian in the form

$$H = \sum_{\mathbf{k}\neq 0}{}' E(k)\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + \text{constant}$$

and show that

$$E(k) = \left[(\frac{\hbar^2 k^2}{2m} + un)^2 - (un)^2 \right]^{1/2} .$$

(ii) The fraction of bosons *not* in the condensate is

$$f = N^{-1} \sum_{\mathbf{k}\neq 0}{}' \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle .$$

Show that

$$f = \frac{\kappa_0^d}{n} I_d$$

where κ_0 is the characteristic wavevector associated with the interaction strength, defined by $\kappa_0^2 = 2mun/\hbar^2$, and

$$I_d = \frac{1}{(2\pi)^d} \int_0^\infty d^d k \frac{1}{2} \left[\frac{1+k^2}{(k^4+2k^2)^{1/2}} - 1 \right].$$

Discuss the convergence of this integral in dimensions $d = 1$, $d = 2$ and $d = 3$.

Qu.3 Consider the ground state of a one-dimensional, non-interacting system of spinless fermions. Let $a^\dagger(x)$ and $a(x)$ be the creation and annihilation operators for a fermion at the point x, so that the density operator is

$$n(x) = a^\dagger(x) a(x).$$

Show that the density-density correlation function has the form

$$\langle n(x) n(0) \rangle = \langle n \rangle^2 (1 - \frac{\sin^2(k_F x)}{(k_F x)^2}) + \langle n \rangle \delta(x)$$

where $\langle n \rangle$ is the mean density, and k_F is the Fermi wavevector.

These oscillations in the density-density correlation function are known as Friedel oscillations, and are present in any number of dimensions.

Qu.4 The intention in this question is to guide you through the exact solution of an interacting many-body problem, the *transverse field Ising model* in one space dimension. The solution uses two operator transformations – the Jordan-Wigner transformation and the Bogoluibov transformation – which are useful in many other contexts.

Consider a one-dimensional lattice with site-label m. Let σ^α, for $\alpha = x, y, z$ be the usual Pauli spin operators. The Hamiltonian for the one-dimensional transverse field Ising model is

$$H = -\Gamma \sum_m \sigma_m^z - J \sum_m \sigma_m^x \sigma_{m+1}^x .$$

(i) Discuss what the ground state would be as a function of J/Γ if σ^x and σ^z were components of a classical unit vector.

(ii) Let $a_m^\dagger$ and a_m be (spinless) fermion creation and annihilation operators. Show that these fermion operators can be written in terms of Pauli raising and lowering operators, $\sigma^\pm = (1/2)(\sigma_x \pm i\sigma_y)$, as

$$a_m = \exp(i\pi \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^-) \sigma_m^-$$

and

$$a_m^\dagger = \exp(-i\pi \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^-) \sigma_m^+ .$$

Show also that $a_m^\dagger a_m = (1 + \sigma_m^z)/2$

(iii) Write down expressions for $\sigma_m^\pm$ and σ_m^z in terms of the fermi operators. These constitute the Jordan-Wigner transformation.

(iv) Use these transformations to write H in terms of fermion operators.

(v) Use a Fourier transform and a Bogoluibov transformation to diagonalise the Hamiltonian. You should obtain

$$H = -\sum_k [(\Gamma + J\cos(k))(\alpha_k^\dagger \alpha_k + \alpha\dagger_{-k}\alpha_{-k} - 1) + iJ\sin(k)(\alpha_k^\dagger \alpha_{-k}^\dagger + \alpha_k \alpha_{-k})]$$

after the Fourier transformation alone, and

$$H = \sum_k \epsilon(k)(2c_k^\dagger c_k - 1)$$

after both transformations, with $\epsilon(k)^2 = \Gamma^2 + J^2 + 2\Gamma J\cos(k)$, where $\alpha^\dagger$, α, $c_k^\dagger$ and c_k are fermion creation and annihilation operators.

(vi) Hence show that the ground-state expectation value, $\langle \sigma_m^z \rangle$, is given by

$$\langle \sigma_m^z \rangle = \frac{1}{\pi} \int_0^\pi dk \frac{\Gamma + J\cos(k)}{\epsilon(k)} .$$