Further Quantum Mechanics: Problem Set 3.

Trinity term weeks 2-3

Questions 1 and 2 are adapted from Shankar. † denotes relatively advanced material.

Qu 1. A system has n non-degenerate single-particle states. These states account for all the degrees of freedom of the particle – position and (if relevant) spin. The states are occupied by a number p of similar non-interacting particles, without any constraints except for those arising from symmetry under particle exchange. How many distinct many-particle states are there for the p-particle system in the following cases:

(i) if the particles are distinguishable;

(ii) if the particles are identical fermions and $p = n$;

(iii) if the particles are identical bosons and $p = n = 2$;

(iv) if the particles are identical bosons and $p = n = 3$?

Qu 2. Consider two identical, non-interacting particles, each of mass m, with coordinates x_1, x_2. They are confined to a box of length L, so that $0 \leq x_1, x_2 \leq L$.

Suppose the particles are identical spin-zero bosons. What is the wavefunction $\psi(x_1, x_2)$ if the system is in an eigenstate with total energy: (i) $\hbar^2\pi^2/mL^2$, or (ii) $5\hbar^2\pi^2/2mL^2$?

Suppose alternatively that the particles are identical spin-half fermions: for the same values of total energy write down a complete set of possible wavefunctions, as products of space and spin factors.

Qu 3. He: ionisation energy. The ionization energy is the energy required to remove one electron from an atom or ion in its ground state, leaving it in the ground state of the next higher ionization stage. (The energy is often quoted in eV and referred to as an ionization potential.)
If we make the (poor) approximation of ignoring the electron-electron repulsion altogether, what value (in eV) is obtained for the ionization potential of the ground state in helium? How much additional energy would then be required to remove the second electron? Assuming these estimates have been made as accurately as reasonably possible within their respective assumptions, state the degree of accuracy of each of these two results (i.e. how close they may be expected to be to the true first and second ionization energies for helium.)

To do better, use the variational method. Using hydrogen-like wavefunctions for both electrons: $\psi_{1s} = \sqrt{Z^3/\pi a_0^3}\,\exp(-Zr/a_0)$, where Z is an adjustable parameter distinct from the nuclear charge, show that

$$\langle \frac{1}{r_1} + \frac{1}{r_2} \rangle = \frac{2Z}{a_0}.$$

Using also $\langle p_1^2 + p_2^2 \rangle = 2Z^2\hbar^2/(a_0^2)$ and $\langle 1/r_{12} \rangle = 5Z/(8a_0)$, show that the mean energy as a function of Z is given in terms of the Rydberg R by

$$E(Z) = -2R\left(4Z - Z^2 - 5Z/8\right).$$

Complete the variational procedure, and hence obtain an upper limit for the ground state energy, and a lower limit for the ionization energy of helium.

Qu 4. Two of the energy levels in Helium have the standard notation $1s2s\,{}^1S_0$ and $1s2s\,{}^3S_1$. Explain every part of this notation. The difference in the notation for the two levels stresses a difference in the spin part of the wavefunction, but the spatial part is also different: what is the important distinguishing feature between the two spatial wavefunctions?

Qu 5.† Recall Qu. 7.6 from Prof Blundell's problem sets, which reads as follows.

A is a beam of atoms with spin-1/2 with the spin alined along the $+x$-axis. B is a beam of similar unpolarised atoms. A and B are separately passed through a Stern-Gerlach apparatus aligned along z. In each case you get two emerging beams coming out of the Stern-Gerlach experiment. Is there any difference between the two cases? If so, how could you detect that experimentally?

Write down density matrices in the z-basis to describe the beams A and B before passage through the Stern-Gerlach apparatus. Use these density matrices to evaluate $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_z \rangle$ for each beam.