Further Quantum Mechanics: Problem Set 2.

Easter vacation and Trinity term weeks 1 – 2

Qns. 3 – 8 are taken from the book by Binney and Skinner. † denotes relatively advanced material.

Qu. 1 (*From 2013 A3 paper.*) The electron in a hydrogen-like ion has a Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$$

and eigenstates $|n, l, m\rangle$.
(a) Explain the origin of each term in the Hamiltonian, and the meaning of each of the quantum numbers n, l and m. On which quantum number does the energy depend? For $n = 2$, what values may l and m take?
(b) By considering the electric dipole selection rules or otherwise, identify which of the matrix elements $\langle 2, l', m'|z|2, l, m\rangle$ are non-zero.
(c) A small static electric field of strength $\mathcal{E}$ is applied in the z direction. Write down the perturbation Hamiltonian. Calculate its non-zero matrix elements for the basis of states with $n = 2$.
(d) Identify the linear combinations of the $n = 2$ states that diagonalise the perturbation Hamiltonian and calculate the energy shifts. Hence sketch the $n = 2$ energy levels before and after the application of the perturbation. In each case, label the eigenstates and give the magnitude of any energy differences.

[The integral $\int_0^\infty \rho^n e^{-\rho}\mathrm{d}\rho = n!$. The normalised states $|n, l, m\rangle = u_n^l(r)\mathrm{Y}_l^m(\theta, \phi)$, with

$$u_2^0(r) = \frac{2e^{-r/2a_Z}}{(2a_Z)^{3/2}}\left(1 - \frac{r}{2a_Z}\right), \qquad u_2^1(r) = \frac{e^{-r/2a_Z}}{\sqrt{3}(2a_Z)^{3/2}}\frac{r}{a_Z}, \qquad a_Z = \frac{4\pi\epsilon_0\hbar^2}{Zme^2},$$

and $\mathrm{Y}_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$, $\quad \mathrm{Y}_1^0(\theta, \phi) = \sqrt{\frac{6}{8\pi}}\cos\theta$, $\quad \mathrm{Y}_1^{\pm1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$.]

Qu. 2.† State and prove the *variational theorem* in quantum mechanics.

Use this theorem to prove that in a one-dimensional system an attractive potential always has a bound state, by taking the following steps. We define an attractive potential $V(x)$ to be one that has finite range a and is negative on average, meaning that

$$V(x) = 0 \quad \text{for} \quad |x| \geq a \qquad \text{and} \qquad \int_{-a}^{a} \mathrm{d}x\, V(x) = -v_0$$

with $v_0 > 0$. Consider a function $f(y)$ that has its maximum at $x = 0$, is non-negative, and approaches zero for $y \to \pm\infty$, with the properties

$$\int_{-\infty}^{\infty} \mathrm{d}y |f(y)|^2 = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}y \left|\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right|^2 = 1\,.$$

For the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$$

use the (unnormalised) trial wavefunction $\psi(x) = f(x/\lambda)$ with variational parameter λ to show that

$$\langle H\rangle \equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}x\, \psi^*(x)\hat{H}\psi(x)}{\int_{-\infty}^{\infty} \mathrm{d}x\, |\psi(x)|^2} < 0$$

for sufficiently large λ. How is the situation different for a three-dimensional system?

Qu 3. Problem 7.1 from Prof Blundell's lecture course is repeated with some modifications below. If you have already attempted it, explain how the approach you used fits within the more general understanding you should now have of time-dependent problems in quantum mechanics. If you have not attempted it previously, you should do so.

In the β decay H^3 (1 proton + 2 neutrons in the nucleus) $\rightarrow$ $(He^3)^+$ (2 protons + 1 neutron in the nucleus), the emitted electron has a kinetic energy of 16 keV. We will consider the effects on the motion of the atomic electron, i.e. the one orbiting the nucleus, which we assume is initially in the ground state of H^3.
Show by a brief justification that the perturbation is sudden, by considering the location of the emitted electron at a time around $\tau = 5 \times 10^{-17}$ s after emission. How does τ compare with the time-scale on which the wavefunction changes?
Show that the probability for the electron to be left in the ground state of $(He^3)^+$ is $2^3(2/3)^6 \simeq 0.7$.

Qu 4. At early times ($t \sim -\infty$) a harmonic oscillator of mass m and natural angular frequency ω is in its ground state. A perturbation $\delta H = \mathcal{E} x \mathrm{e}^{-t^2/\tau^2}$ is then applied, where $\mathcal{E}$ and τ are constants.
What is the probability according to first-order theory that by late times the oscillator transitions to its second excited state, $|2\rangle$?
Show that to first order in δH the probability that the oscillator transitions to the first excited state, $|1\rangle$, is

$$P = \frac{\pi \mathcal{E}^2 \tau^2}{2m\hbar\omega} \mathrm{e}^{-\omega^2\tau^2/2},$$

Plot P as a function of τ and comment on its behaviour as $\omega\tau \rightarrow 0$ and $\omega\tau \rightarrow \infty$.

Qu 5. Write down the selection rules for radiative transitions in the electric dipole approximation. Draw an energy level diagram for hydrogen (use the vertical direction for energy, and separate the states horizontally by angular momentum ℓ). Show how the selection rules apply to hydrogen by marking allowed transitions on your diagram.

Qu 6. With $|nlm\rangle$ a stationary state of hydrogen, which of these matrix elements is non-zero?

$$\begin{array}{ccc} \langle 100|z|200\rangle & \langle 100|z|210\rangle & \langle 100|z|211\rangle \\ \langle 100|z|300\rangle & \langle 100|z|310\rangle & \langle 100|z|320\rangle \\ \langle 100|x|200\rangle & \langle 100|x|210\rangle & \langle 100|x|211\rangle \end{array}$$

Qu 7. With $|nlm\rangle$ a stationary state of hydrogen, and given that

$$Y_{10}(\theta,\phi) = \sqrt{\frac{6}{8\pi}}\cos\theta \quad Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{i\phi} \quad Y_{1-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-i\phi},$$

show that

$$\begin{aligned} \langle 100|(x-iy)|211\rangle &= -\sqrt{2}\langle 100|z|210\rangle \\ \langle 100(x-iy)|21-1\rangle &= 0. \end{aligned}$$

Write down the values of $\langle 100|(x+iy)|21-1\rangle$ and $\langle 100|(x+iy)|211\rangle$ and hence show that with

$$|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|211\rangle - |21-1\rangle),$$

$\langle 100|x|\psi\rangle = -\langle 100|z|210\rangle$. Explain the physical significance of this result.

Qu 8. Derive the selection rules

$$\begin{aligned} \langle n'l'm'|x_+|nlm\rangle &= 0 \quad \text{unless} \quad m' = m+1 \\ \langle n'l'm'|x_-|nlm\rangle &= 0 \quad \text{unless} \quad m' = m-1. \end{aligned}$$

where $x_\pm = x \pm iy$. From this selection rule one infers that when the atom sits in a magnetic field along the z axis and the spectrometer looks along the z axis, the detected photons will be circularly polarised. Show that linearly polarised photons *can* be detected from an atom that's in a magnetic field.
From the above rules it might be argued that photons emitted along the z axis will be circularly polarised even in the absence of a magnetic field. Why is this argument bogus?