Further Quantum Mechanics: Problem Set 1.

Hilary Term, weeks 6 – 8

Problems 2-8 are taken from the book by Binney and Skinner. † denotes relatively advanced material.

Qu 1.† A particle with mass m, charge q and momentum $\hat{\mathbf{p}}$ moves in a magnetic field described by the vector potential $\mathbf{A}$. The Hamiltonian is

$$H = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 \,.$$

The wavefunction $\psi(\mathbf{r}, t)$ satisfies the time-dependent Schrödinger equation with this Hamiltonian, and the probability density $\rho \equiv |\psi(\mathbf{r}, t)|^2$ obeys the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} \,.$$

Show from the time-dependent Schrödinger equation that

$$\mathbf{j} = \frac{\hbar}{2im}\{\psi^* \nabla \psi - \psi \nabla \psi^*\} - \frac{q}{m}|\psi|^2 \mathbf{A} \,.$$

The particle moves in the x-y plane with a uniform magnetic field in the z-direction, represented by

$$\mathbf{A} = \frac{B}{2}(-y, x, 0).$$

Show that the function

$$\varphi(x, y) = \exp(-[x^2 + y^2]/4\ell_{\mathrm{B}}^2)$$

is an eigenfunction for a suitably chosen value for ℓ_{B}. Find this value of ℓ_{B} and the energy. Evaluate $\mathbf{j}$ for this state. Discuss the physical significance of your results.

Qu 2. The Hamiltonian of a two-state system can be written

$$H = \begin{pmatrix} A_1 + B_1\epsilon & B_2\epsilon \\ B_2\epsilon & A_2 \end{pmatrix},$$

where all quantities are real and ϵ is a small parameter. To first order in ϵ, what are the allowed energies in the cases (a) $A_1 \neq A_2$, and (b) $A_1 = A_2$?
Obtain the exact eigenvalues and recover the results of perturbation theory by expanding in powers of ϵ.

Qu 3. The harmonic oscillator with mass m and angular frequency ω is perturbed by $\delta H = \epsilon x^2$. (a) What is the exact change in the ground-state energy? Expand this change in powers of ϵ up to order ϵ^2. (b) Show that the change given by first-order perturbation theory agrees with the exact result to $\mathcal{O}(\epsilon)$ (c) Show that the first-order change in the ground state is $|b\rangle = -(\epsilon\ell^2/\sqrt{2}\hbar\omega)|E_2\rangle$. (d) Show that second-order perturbation theory yields an energy change $E_c = -\epsilon^2\hbar/4m^2\omega^3$ in agreement with the exact result.

Qu 4. The harmonic oscillator is perturbed by $\delta H = \epsilon x$. Show that the perturbed Hamiltonian can be written

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 X^2 - \frac{\epsilon^2}{\omega^2}\right),$$

where $X = x + \epsilon/m\omega^2$ and hence deduce the exact change in the ground-state energy. Interpret these results physically.
What change in energy eigenvalues does first-order perturbation theory give? From perturbation theory determine the coefficient b_1 of the unperturbed first-excited state in the perturbed ground state. Discuss your result in relation to the exact ground state of the perturbed oscillator.

Qu 5. The harmonic oscillator is perturbed by $\delta H = \epsilon x^4$. Show that the first-order change in the energy of the nth excited state is

$$\delta E = 3(2n^2 + 2n + 1)\epsilon\left(\frac{\hbar}{2m\omega}\right)^2.$$

Hint: express x in terms of $A + A^\dagger$.

Qu 6. The infinite square-well potential $V(x) = 0$ for $|x| < a$ and ∞ for $|x| > a$ is perturbed by the potential $\delta V = \epsilon x/a$. Show that to first order in ϵ the energy levels of a particle of mass m are unchanged. Show that even to this order the ground-state wavefunction *is* changed to

$$\psi_1(x) = \frac{1}{\sqrt{a}}\cos(\pi x/2a) + \frac{16\epsilon}{\pi^2 E_1\sqrt{a}}\sum_{n=2,4,}(-1)^{n/2}\frac{n}{(n^2-1)^3}\sin(n\pi x/2a),$$

where E_1 is the ground-state energy. Explain physically why this wavefunction does not have well-defined parity but predicts that the particle is more likely to be found on one side of the origin than the other. State with reasons but without further calculation whether the second-order change in the ground-state energy will be positive or negative.

Qu 7. An atomic nucleus has a finite size, and inside it the electrostatic potential $\Phi(r)$ deviates from $Ze/(4\pi\epsilon r)$. Take the proton's radius to be $a_p \simeq 10^{-15}$m and its charge density to be uniform. Then treating the difference between Φ and $Ze/(4\pi\epsilon_0 r)$ to be a perturbation on the Hamiltonian of hydrogen, calculate the first-order change in the ground-state energy of hydrogen. Why is the change in the energy of any P state extremely small? Comment on how the magnitude of this energy shift varies with Z in hydrogenic ions of charge Z. Hint: exploit the large difference between a_p and a_0 to approximate the integral you formally require.

Qu 8. A particle of mass m moves in the potential $V(x, y) = \frac{1}{2}m\omega^2(x^2 + y^2)$, where ω is a constant. Show that the Hamiltonian can be written as the sum $H_x + H_y$ of the Hamiltonians of two identical one-dimensional harmonic oscillators. Write down the particle's energy spectrum. Write down kets for two stationary states in the first-excited level in terms of the stationary states $|n_x\rangle$ of H_x and $|n_y\rangle$ of H_y. Show that the nth excited level is $n + 1$ fold degenerate.
The oscillator is disturbed by a small potential $H_1 = \lambda xy$. Show that this perturbation lifts the degeneracy of the first excited level, producing states with energies $2\hbar\omega \pm \lambda\hbar/2m\omega$. Give expressions for the corresponding kets.
The mirror operator M is defined such that $\langle x, y|M|\psi\rangle = \langle y, x|\psi\rangle$ for any state $|\psi\rangle$. Explain physically the relationship between the states $|\psi\rangle$ and $M|\psi\rangle$. Show that $[M, H_1] = 0$. Show that $MH_x = H_yM$ and thus that $[M, H] = 0$. What do you infer from these commutation relations?

Qu 9. The Hamiltonian for a hydrogen atom in a uniform magnetic field with flux density B along z is approximately

$$H = \frac{\hat{\mathbf{p}}^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar e B}{2m}(L_z + \sigma_z)$$

where $\hbar L_z$ and $(\hbar/2)\sigma_z$ are operators representing respectively the z-components of the orbital and spin angular momentum of the electron.

Discuss how the terms involving B arise. What is the justification for omitting a term in B^2?

What quantum numbers label eigenstates of this Hamiltonian?

Draw a diagram showing the energies of a representative selection of these eigenstates as a function of magnetic field.