Random planar curves and conformal field theory. Case for Support.

1 Previous track record

The PI began research in the mathematics of particle theory before switching to applications of quantum field theory to statistical mechanics and condensed matter. In the late 1980s, in a series of papers, he developed the fundamentals of Boundary Conformal Field Theory (BCFT) [1,2,3] (for which he received the 2004 Lars Onsager Prize of the APS.) While it was initially aimed at 2d critical behaviour, this work has also had important impacts on both string theory and condensed matter physics, as well as the mathematics of CFT. In 1990 the PI applied these ideas to percolation theory, and, inspired by work of Langlands et al [15], conjectured [4] a formula for the probability that a critical cluster spans between two disjoint segments of the boundary of a simply connected region. This became known as 'Cardy's formula'. In the late 1990s, Schramm, Lawler and Werner [16] developed the theory of stochastic Loewner evolution (SLE) partly in order to explain this formula, although it was finally proved using another method by Smirnov [17] (for which he received the Clay Medal). During this period the PI produced several other formulae for percolation and related problems [5,6,7,8,9,10,11] using methods of CFT, most of which have yet to be proved. In the last few years, among other things, he has been involved, with others, in understanding the relation between CFT and SLE [12,13,14].

This has been one of the most successful examples of cross-fertilisation between theoretical physics and pure mathematics. The PI has been regularly invited to give talks at international mathematics conferences and workshops. At present he is the only senior researcher in the UK working in this rapidly developing area.

Oxford University provides a unique environment within the UK to study these types of problem, both in the subdepartment of Theoretical Physics, where several groups work on applications of CFT to string theory and condensed matter physics, and in Mathematics, with very strong groups in string theory, analytic geometry and stochastic analysis.

1.0.1 PI's publications relevant to this proposal

1. Conformal Invariance and Surface Critical Behavior, *Nuclear Physics* **B240 [FS12]**, 514-532, 1984.

2. Boundary Conditions, Fusion Rules and the Verlinde Formula, *Nucl. Phys. B* **324**, 581, 1989.

3. Boundary Conformal Field Theory, to appear in *Encyclopedia of Mathematical Physics*, J.-P. Françoise, G. Naber and T.S. Tsun, eds. (Elsevier, 2005.)

4. Critical Percolation in Finite Geometries, *J. Phys. A* **25**, L201, 1992.

5. Mean Area of Self-Avoiding Loops, *Phys. Rev. Lett.* **72**, 1580, 1994.

6. Geometrical Properties of Loops and Cluster Boundaries, in ‘Fluctuating Geometries in Statistical Mechanics and Field Theory’, Les Houches Summer School 1994 (North-Holland, 1996.).

7. The Number of Incipient Spanning Clusters in Two-Dimensional Percolation, *J. Phys. A* **31**, L105, 1998.

8. Linking numbers for self-avoiding loops and percolation: application to the spin quantum Hall transition, *Phys. Rev. Lett.* **84**, 3507, 2000.

9. Exact Scaling Functions for Self-Avoiding Loops and Branched Polymers, *J. Phys. A* **34**, L665, 2001.

10. Exact Results for the Universal Area Distribution of Clusters in Percolation, Ising and Potts Models (with R. M. Ziff), *J. Stat. Phys.* **110**, 1, 2003.

11. Crossing Formulae for Critical Percolation in an Annulus. *J. Phys. A* **35**, L565, 2002.

12. Stochastic Loewner Evolution and Dyson's Circular Ensembles, *J. Phys. A* **36**, L379, 2003; erratum *J. Phys. A* **36**, 12343, 2003.

13. Calogero-Sutherland Model and Bulk-Boundary Correlations in Conformal Field Theory, *Phys. Lett. B* **582**, 121, 2004.

14. SLE for theoretical physicists, review article, Ann. Phys. 318(1), 81, 2005.

1.0.2 Other references.

15. R. Langlands, P. Pouliot and Y. Saint-Aubin, *Conformal invariance and two-dimensional percolation,* Bulletin of the AMS **30**, 1, 1994 (math.MP/9401222).

16. O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees,* Israel J. Math. **118**, 221, 2000 (math.PR/9904022); W. Werner, *Random planar curves and Schramm-Loewner evolutions,* in *Ecole d'Eté de Probabilités de Saint-Flour XXXII (2002)*, Springer Lecture Notes in Mathematics **1180**, 113, 2004 (math.PR/0303354); G.F. Lawler, *Conformally invariant processes in the plane,* ICTP Lecture Notes **17**, 305, 2004 (http://www.ictp.trieste.it/~pub_off/lectures/vol17.html); *Stochastic Loewner Evolution,* to appear in *Encyclopedia of Mathematical Physics*, J.-P. Françoise, G. Naber and T.S. Tsun, eds. (Elsevier, 2005.) (http://www.math.cornell.edu/ lawler/encyclopedia.ps) *Conformally Invariant Processes in the Plane*, book, (American Math. Soc., 2005).

17. S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits,* C. R. Acad. Sci. Paris Sér. I Math. **333(3)**, 239, 2001.

2 Description of the proposed work and its context

2.1 Background

Understanding random spatial processes has a long history of study. Many insights have come from physics, but recently there has been much progress on the mathematical side. Many such processes are related to equilibrium measures in statistical mechanics, but equally important from a mathematical point of view are those generated by stochastic processes such as Brownian motion. One of the successes of the current leap forward has been to connect these two. The equilibrium measures are initially discrete, for example the Ising model, random percolation, or self-avoiding walks, all on some fixed lattice with mesh size δ, but the arguments from physics then suggest that it should be possible to make sense of the *scaling limit* $\delta \to 0$. In that limit, physics asserts, such systems correspond to renormalisable (euclidean) quantum field theories, which, from the most pragmatic point of view, are sets of correlation functions $C(r_1, \ldots, r_N)$ – maps from N copies of the base manifold (with coincident points $r_i = r_j$ removed) to the complex numbers, which satisfy certain constraints, expressed through the operator product expansion, as the points approach each other. In the language of physics these correlation functions are often thought of as expectation values of so-called local operators with respect to some measure, but it is in precisely defining these notions that constructive field theory gets stuck. The interesting case of a massless theory is even more difficult from the constructive point of view: it is believed to correspond to a *conformal* field theory, in which a privileged role is played by those local operators, or rather their correlators, which are holomorphic functions of the coordinates.

The new perspective avoids the direct construction of local operators by focusing instead on the random planar (non-crossing, but possibly self-intersecting) curves which characterise the random spatial process. For the Ising model and percolation these are the cluster boundaries; for 2d Brownian motion they might be the frontiers or outer boundaries. The essential steps forward were made in a series of papers by Lawler, Schramm and Werner (LSW). Any such curve γ which connects, for example, two points on the boundary of a simply connected domain $\mathcal{D}$ can be viewed as being 'grown' as a sequence of sets γ_t, with $\gamma_t \subset \gamma_{t'}$ if $t < t'$, by a process called Loewner evolution. Instead of γ_t this focuses on the conformal map g_t, whose existence is guaranteed by the Riemann mapping theorem, and which sends $\mathcal{D} \backslash \gamma_t$ into $\mathcal{D}$ (this needs modifying slightly if γ is not simple.) In the case when $\mathcal{D}$ is the upper half plane, g_t, suitably normalised at infinity, sends the growing tip of γ_t into a unique point a_t on the real axis. Loewner showed that the time evolution $dg_t(z)/dt$ for $z \notin \gamma_t$ obeys a simple ODE which is determined solely by a_t.

In the case of a random planar curve, a_t is a stochastic function. Schramm[16] argued that if the measure on γ is conformally invariant, in a well-defined sense, the only possibility for a_t is to be proportional to a standard 1d Brownian motion: $a_t = \sqrt{\kappa} B_t$, for some $\kappa > 0$. This is called stochastic-, or Schramm-, Loewner evolution (SLE). By direct computations, as well as comparing with known results from 2d Brownian motion, LSW were able rigorously to derive many of the critical indices of 2d critical behaviour, under the assumption that the curves in the lattice models are conformally invariant in the scaling limit. This was shown for site percolation on the triangular lattice by Smirnov, on the basis of his proof of the PI's 1991 conjectured crossing formula.

It has taken theoretical physicists some time to catch up with this explosion of new results. A step forward was made by Bauer and Bernard (and independently Friedrich and Werner) who realised that the martingales of SLE are equivalent to the statement that certain local operators of CFT (associated with the points where the curve is attached to the boundary of the domain) correspond to so-called level 2 null state representations of the Virasoro algebra (this was already conjectured by the PI in 1984.) This gave a direct link between SLE and certain sectors of the CFT, but by no means the complete picture. The PI generalised this to multiple curves, and showed that the corresponding PDEs are of the Calogero-Sutherland type.

Meanwhile among the mathematical activists the emphasis has moved away from SLE itself to more general ways of defining conformally invariant measures on planar curves. For example, conformal restriction asserts that if $\mathcal{D}' \subset \mathcal{D}$, then the measure on γ in $\mathcal{D}$, conditioned to lie in $\mathcal{D} \setminus \mathcal{D}'$, is the same as that induced by the conformal mapping $\mathcal{D} \to \mathcal{D} \setminus \mathcal{D}'$. For SLE, this works only for $\kappa = \frac{8}{3}$, (and on the lattice trivially for self-avoiding walks.) However it can be applied to other situations, such as self-avoiding loops, for which the SLE setup is inappropriate. LSW together with Sheffield have used the measure from conformal restriction to construct other conformally invariant measures on multiple loops (the 'conformal loop ensemble' (CLE)) which are conjectured to give the full scaling limit of other models. This may be termed the 'bottom up' approach. In a parallel development, Schramm and Sheffield have shown that the level lines of a gaussian free field correspond to SLE_4, and then suggested how to get other values of κ: the 'top down' approach.

2.2 Programme and methodology

2.2.1 Overall aims of the project.

Given the current status of the field described above, and the background of the PI, there are two separate but related strands to the proposed project:

- *constructing CFT from SLE and its generalisations* : the aim here is to give a direct and rigorous construction of the correlation function of CFTs, as massless quantum field theories, starting from SLE and its related ideas.
- *convergence of lattice models to SLE and related measures* : the aim is to prove that certain curves in well-known lattice models (such as the Ising model) converge in the scaling limit to SLE with a suitable value of κ.

2.2.2 Programme of work

Constructing CFT from SLE and its generalisations.
Among mathematicians and mathematical physicists, conformal field theory (CFT) is generally conceived of as being developed from a set of axioms, most commonly those systematised by G Segal. These already presuppose a fair amount of structure, in particular how the conformal invariance is encoded in the way the Virasoro algebra acts on states. Theoretical physicists believe that CFT developed in this way does indeed describe 2d massless renormalised quantum field theory, but since such theories have never been mathematically

constructed (except for free fermions) this is merely a pious hope. It is important to provide a deeper basis for CFT, and SLE and its related ideas provide the hope for doing this.

Some progress along these lines has recently been made by the PI with Oxford collaborators Benjamin Doyon (EPSRC postdoc) and Valentina Riva (PDRA). We tentatively identified the stress-energy tensor $T(z)$ of CFT in terms of the spin-2 Fourier component $\int d\theta e^{-2i\theta}$ of the probability that the curve γ intersects a small slit $(z-\epsilon e^{i\theta}, z+\epsilon e^{i\theta})$, in the limit $\epsilon \to 0$, after a rescaling by ϵ^{-2}. In the case when conformal restriction holds, we showed that such quantities satisfy the conformal Ward identities of CFT which underlie the Virasoro algebra. The resulting CFT, however, has central charge $c = 0$. The continuation of this programme we see as proceeding in two stages:

- identifying a complete set of 'local operators' in the conformal restriction case. What is meant here is a set of correlation functions which close under the operator product expansion as points approach each other in a pairwise fashion. The above Ward identity argument generalises to the case when the curve is constrained to pass around a set of marked points $\{\zeta_j\}$ in some prescribed fashion. By taking the limit, say, as $\zeta_{12} \equiv \zeta_1 - \zeta_2 \to 0$ we can in principle obtain an expansion in increasing powers of ζ_{12}. The leading term in fact goes like $|\zeta_{12}|^{2/3}$, reflecting the non-trivial fractal dimension of the curve. In CFT, this couples to the 'energy density' operator. Other higher powers define, in principle, the correlators of other higher-dimensional operators. What is needed is to show that if we take 2 such pairs of points (ζ_1, ζ_2) and (ζ_3, ζ_4), generate a set of operators as above, and then allow the two pairs to come together, we do not generate anything new. This can be done in principle by analysing the equations one gets from conformal restriction.

- generalising the above to CFTs with central charge $c > 0$. The most promising basis for this is the conformal loop ensemble (CLE) mentioned above. In this construction, a measure on multiple loops is defined through an independent Poisson process of intensity $\propto c$, based on the measure on single loops. These loops of course overlap, but for small enough c they form finite clusters, and by considering only their 'outer boundaries' this defines a measure on unnested multiple self-avoiding loops. A nested set of loops can then be constructed iteratively, by regarding each loop as the boundary of a region in the plane and repeating the above construction. It is conjectured that the measure on self-avoiding nested loops so obtained is the scaling limit of the loops in the lattice $O(n)$, with a suitable relation between c and n (already known in the CFT literature.) The proposal is to use the CLE to derive, first, the conformal Ward identities, then, if this succeeds, a complete set of local operators following the ideas above. The candidate for the stress tensor is, as above, the spin-2 component of the probability that a loop (now *any* loop) intersects a small slit. We already know this works for a single loop, which corresponds to $c = 0$. However, the CLE construction both takes away parts of loops (in considering only the outer boundaries) and adds loops back in (through the nesting procedure). It will be necessary to show that these steps modify the Ward identities only by generating the expected terms proportional to c.

Convergence of lattice models to SLE.
It is known that the measures of SLE and the related CLE can be constructed somewhat

indirectly from 2d Brownian motions. What is not yet shown, except in a few cases (although there is overwhelming numerical evidence,) is that random curves in lattice models converge to these continuum measures. In a famous result, Smirnov showed that this is the case for site percolation on the triangular lattice, and SLE_6. He did this by deriving the PI's 1991 crossing formula, and showing that it holds in an arbitrary domain. It turns out that this is enough to prove that the driving term in the Loewner equation is $\sqrt{6}$ times a standard Brownian motion. The essential element in Smirnov's proof was to identify a quantity on the lattice which converges to a holomorphic function in the limit as the mesh size $\delta \to 0$.

It turns out that similar holomorphic object can be defined within SLE_κ for other values of κ. One such was described above for $\kappa = \frac{8}{3}$: the spin-2 component of the probability that the curve intersects a short slit of length ϵ, scaled with a suitable power of ϵ. For other values of the spin s, a simple SLE calculation shows that holomorphicity holds if $\kappa = 8/(s+1)$. This suggests that we try to define similar lattice objects and to prove that, as $\delta \to 0$, they approach holomorphic quantities. There are specific values of s for which this programme seems feasible:

- $s = 1$: this should correspond to $\kappa = 4$, that is $c = 1$ and therefore the gaussian free field. It has already been shown by Schramm and Sheffield that the level lines of this field converge to SLE_4. However there are other models which are believed by physicists to have the same scaling limit. One is the discrete gaussian model, in which the variables are restricted to be integers. For this model, the spin-1 object should just be the current carried along the level lines. The central problem is to show this is approximately holomorphic. This may follow from the detailed balance condition, which if naively linearised gives Laplace's equation. It would be sufficient to prove that this linearisation is valid in the scaling limit.

- $s = \frac{1}{2}$: this should correspond to $\kappa = \frac{16}{3}$, when the SLE curve is conjectured to describe the cluster boundaries of the Fortuin-Kasteleyn clusters of the Ising model (considered as the 2-state Potts model). In fact one can show that these boundaries are just the world-lines of the famous Ising fermion of Onsager's solution. This is known to be 'free', that is satisfy simple linear equations, even on the lattice. These should lead to a suitably holomorphic object in the scaling limit.

- other fractional values of s: it is known that the CFTs which are conjectured to give the scaling limits for other values of Q in the q-state Potts model contain 'parafermionic' currents with fractional spin s. Similar objects, with spin $s = 1/(m-1)$, also occur in the so-called minimal models of CFT, the series of unitary theories labelled by integers m. Lattice realisations of these are believed to be the RSOS models of Andrews, Baxter and Forrester, in which the heights of the discrete gaussian model are restricted to the values $(1, 2, \ldots, m-1)$. In these models it is simple to identify candidate curves whose scaling limit is given by SLE: the challenge is to show that the spin-s component of the probability they intersect a small slit is holomorphic in the scaling limit. This appear to be more difficult than for the Ising case $m = 3$ when it satisfies a simple linear equation. However, once again by using detailed balance, one can hope to get a non-linear equation which simplifies in the scaling limit.