The Ubiquitous "c": from the Stefan-Boltzmann law to Quantum Information

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Stefan's Law[1]

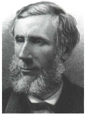
Energy density of black body radiation $u = (4\pi/c)\sigma T^4$ where $\sigma \approx 5.67 \times 10^{-8} \mathrm{Js^{-1}m^{-2}K^{-4}}$

[1] J. Stefan, Über die Beziehung zwischen der Wärmestrahlung und der Temperatur, *Wiener Berichte*, **79**, 391, 1879.

Stefan's Law

Energy density of black body radiation $u = (4\pi/c)\sigma T^4$ where $\sigma \approx 5.67 \times 10^{-8} \mathrm{Js^{-1}m^{-2}K^{-4}}$

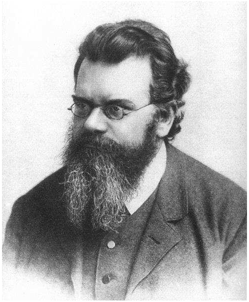
- based on experimental measurements by J. Tyndall[1]



[1] better known in some circles as an Alpine mountaineer!

Boltzmann's Derivation[2]

- imagine a piston is moved by radiation pressure $P = u/3$



$$
\begin{aligned}
TdS &= d(uV) + PdV \\
&= u'(T)VdT + \tfrac{4}{3}udV \\
\frac{\partial}{\partial V}\left(\frac{V}{T}u'(T)\right) &= \frac{\partial}{\partial T}\left(\frac{4u}{3T}\right) \\
\frac{u'(T)}{T} &= \frac{4u}{T^2} \\
u(T) &\propto T^4
\end{aligned}
$$

- "A true pearl of theoretical physics" [Lorentz, 1907]

[2] L. Boltzmann, Ableitung des Stefan'schen Gesetzes. . ., *Wiedemann's Annalen*, **22**, 291, 1884.

Why Boltzmann was lucky

- this is a *quantum* phenomenon ($\sigma = \pi^4 k^4 / 60\hbar^3 c^2$) yet $P = u/3$ was a result of *classical* electrodynamics
- in modern terms, it relies on the assumption that the energy-momentum tensor $T_{\mu\nu}$ is *traceless*, so

$$u = \langle T_{00} \rangle = \sum_{j=1}^{3} \langle T_{jj} \rangle = 3P$$

- however there are plenty of field theories where $T^{\mu}_{\mu} = 0$ classically but not at the quantum level!
  - quantum electrodynamics with vacuum polarization effects
  - non-Abelian gauge theories

- in general $T^{\mu}_{\mu} \propto \beta(g)$ – it vanishes at RG fixed points
- we expect the generalised Stefan-Boltzmann law to hold at *quantum critical points* with $z = 1$, *i.e.* $\omega \sim v|k|$
- for $d = 1$ (*e.g.* free fermions at finite density, Luttinger liquids, quantum Hall edge states, and many critical quantum spin chains)

$$u = \frac{\pi c}{6\hbar v}(kT)^2$$

where

$$c = \begin{cases} 1 & : \text{ single free boson} \\ N & : \ N \text{ free bosons} \\ \frac{1}{2} & : \text{ a spinless fermion} \end{cases}$$

- in general c is fractional: the *conformal anomaly number* of the corresponding conformal field theory (CFT)

Why is the conformal anomaly anomalous?

- go to 2-dimensional euclidean space: quantum criticality in $d=1 \Leftrightarrow$ classical criticality in $d=2$
- use ‘complex co-ordinates’

$$z = x_1 + ix_2, \qquad \bar{z} = x_1 - ix_2$$

- *e.g.* free boson, action

$$S = \int (\nabla\phi)^2 d^2x \propto \int (\partial_z\phi)(\partial_{\bar{z}}\phi) d^2z$$

- classically, the only non-zero components of the stress tensor are

$$T(z) = (\partial_z\phi(z,\bar{z}))^2 \qquad \overline{T} = (\partial_{\bar{z}}\phi(z,\bar{z}))^2$$

- classically, under a conformal transformation $z \to f(z)$

$$S \to S \quad \text{and} \quad T(z) = f'(z)^2\, T(f(z))$$

- but, taking into account the fluctuations, this definition of T is *divergent*: instead define

$$T(z) = \lim_{\delta \to 0} \left[\partial_z \phi(z + \tfrac{\delta}{2}) \partial_z \phi(z - \tfrac{\delta}{2}) - \frac{1}{2\delta^2} \right]$$

- this subtraction does not commute with the conformal transformation: instead

$$T(z) = f'(z)^2 \, T(f(z)) - \frac{c}{12} \{f, z\} \qquad (*)$$

  where $\{f, z\} = (f'''f' - \frac{3}{2}f''^2)/f'^2$, and in this case $c = 1$
- more generally this is the only possible form for the anomaly
- from $(*)$ emerge the different physical manifestations of c

Stefan-Boltzmann law in 1+1 dimensions[3]

- finite temperature density matrix $e^{-\hat{H}/kT}$ $\Leftrightarrow$ euclidean path integral on cylinder circumference $\beta = \hbar/kT$
- conformal mapping plane $\rightarrow$ cylinder: $f(z) = (\beta/2\pi)\log z$

$$\langle T(z)\rangle_{\text{plane}} = 0 = f'(z)^2\,\langle T(f(z))\rangle_{\text{cylinder}} - \frac{c}{12}\{f, z\}$$

giving

$$u = \langle T_{00}\rangle_{\text{cylinder}} = \frac{\pi c}{6\hbar}(kT)^2$$

[3] Blöte, JC, Nightingale 1986; Affleck 1986

The conformal periodic table[4]

- in the operator formulation of CFT, moments $L_n = \oint z^{n+1} T(z) dz$ satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12} c\, n(n^2-1)\delta_{n,-m}$$

- the scaling fields fall into highest weight representations of this algebra with the eigenvalue of L_0 giving the scaling dimensions x ($\rightarrow$ critical exponents, by scaling)
- for $c < 1$ there is a countable list of possibilities $c = 1 - 6/m(m+1)$, $m = 3, 4, 5, \ldots$ and for each m only $\frac{1}{2}m(m-1)$ possible values for the scaling dimensions x:

[4] Belavin, Polyakov, Zamolodchikov 1984; Friedan, Qiu, Shenker 1984

c	Scaling dimensions	Universality class
$\frac{1}{2}$	$0, \frac{1}{8}, 1$	Critical Ising
$\frac{7}{10}$	$0, \frac{3}{80}, \frac{1}{10}, \frac{7}{16}, \frac{3}{5}, \frac{3}{2}$	Tricritical Ising
$\frac{4}{5}$	$0, \frac{1}{40}, \frac{1}{15}, \frac{1}{8}, \frac{2}{5}, \frac{21}{40}, \frac{2}{3}, \frac{7}{5}, \frac{13}{8}, 3$	Tetracritical Ising
$\frac{4}{5}$	$0, \frac{1}{15} \times 2, \frac{2}{5}, \frac{2}{3} \times 2, \frac{7}{5}, 3$	Critical 3-state Potts
$\vdots$	$\vdots$	$\vdots$

- these correspond to well-known universality classes
- demanding the consistency of the CFT on the torus (modular invariance[5]) dictates exactly *which* scaling dimensions appear in a given CFT
- this partially realises (at least in $d = 2$) the RG programme of enumerating *all* universality classes

[5] JC 1986

c and Entanglement Entropy[6]

- c also quantifies the degree of *quantum entanglement* in the ground state of 1+1-dimensional systems near a quantum critical point.
- simplest scenario: a 1d infinite system in divided in two halves A and B
- A's reduced density matrix

$$\rho_A = \mathrm{Tr}_B \, |0\rangle\langle 0|$$

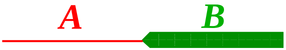
- Rényi entropies $S_A{}^{(n)} = (1-n)^{-1} \log \mathrm{Tr}_A \rho_A{}^n$ measure the degree of entanglement between A and B

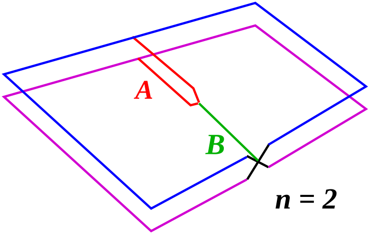
[6] Holzhey, Larsen, Wilczek 1994; Calabrese, JC 2004

Path integral representation

$$\mathrm{Tr}_A\, \rho_A{}^n = \frac{Z_n}{Z_1^n}$$

where Z_n is the path integral (partition function in 2d) on an n-sheeted surface $\mathcal{R}_n$:





- $\mathcal{R}_n$ is related to a single sheet by the conformal mapping $f(z) = z^{1/n}$: using (*)

$$\langle T \rangle_{\mathcal{R}_n} = \frac{c(1 - n^{-2})}{12z^2}$$

- this means that the branch point behaves like a scaling field of dimension $x_n = (c/12)(n - n^{-1})$
- close to the critical point

$$S_A^{(n)} \sim (1-n)^{-1} \log\left(\xi^{-x_n}\right) = \frac{c}{12}(1 + n^{-1}) \log \xi$$

- just the the tip of the iceberg of results of this nature!

Other appearances of c

- if we put a CFT on a manifold of Euler character χ and linear size L, free energy

$$F \sim AL^2 + BL - \tfrac{1}{6}c\chi \log L + \cdots$$

- Zamolodchikov's c theorem[7]: "There exists a function $C(g)$ which is decreasing along RG flows and is stationary at RG fixed points, where it equals c"
- the c-theorem sum rule[8]: *e.g.* in a magnetic field at T_c

$$c = 3\pi^2 \left(\frac{\delta}{\delta+1}\frac{H}{k_B T_c}\right)^2 \left.\chi''(q)\right|_{q=0}$$

where $\chi(q)$ is the q-dependent susceptibility

[7] A B Zamolodchikov 1986

[8] JC 1986

. . . at least in 2 (or 1+1) dimensions

is everywhere!