Entanglement Entropy in Extended Quantum Systems

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STATPHYS 23 Genoa

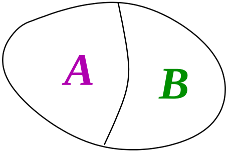
Outline

- A. Universal properties of entanglement entropy near quantum critical points

- B. Behaviour of entanglement and correlation functions after a ‘quantum quench’

A. Quantum Entanglement (Bipartite)

- quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$
- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



- Alice can make observations only in *A*, Bob in the complement *B*
- in general Alice's measurements are entangled with those of Bob

Measuring entanglement

- Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j \, |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$.

- one measure of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$

- if $c_1 = 1$, rest zero, $S = 0$ and $|\Psi\rangle$ is unentangled
- if all c_j equal, $S \sim \log \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$ – maximal entanglement

- equivalently, in terms of Alice's reduced density matrix:

$$\rho_A \equiv \mathrm{Tr}_B \, |\Psi\rangle\langle\Psi|$$

von Neumann entropy

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A$$

- other measures of entanglement exist, but entropy has several nice properties: additivity, convexity, . . .
- in quantum information theory, it gives the efficiency of conversion of partially entangled → maximally entangled states by local operations (Bennet et al)
- it gives the amount of classical information required to specify ρ_A (important for density-matrix RG)

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- it gives a basis-independent way of identifying and characterising quantum phase transitions

In this talk we consider the case when:

- the degrees of freedom of the quantum system are extended over some large region $\mathcal{R}$ in $\mathbb{R}^d$
- the hamiltonian H contains only short-range interactions, e.g:
  - a lattice quantum spin system
  - a UV cut-off quantum field theory
- A is the set of degrees of freedom in some large (compact) subset of $\mathcal{R}$
- the whole system is in a pure state, usually the ground state $|0\rangle$ of H (although it will be useful to consider this as the limit $\beta \to \infty$ of a thermal mixed state with $\rho \propto e^{-\beta H}$)

- since first papers (Osborne and Nielsen 2002; Osterloh *et al* 2002; Vidal *et al* 2003), $>$ 150 papers, mostly on exact or numerical calculations in quantum spin models [Review: (Amico *et al* 2007)]
- in this talk I will focus on the universal properties, which can be extracted from the QFT description of critical behaviour (Calabrese, JC 2004*ff*)

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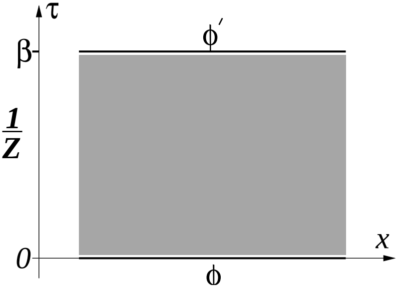
How does S_A depend on the size and geometry of A
and the universality class of the critical behaviour?

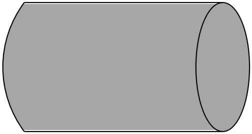
Entanglement entropy from the path integral

"Quantum mechanics is just statistical mechanics in one more dimension" – M E Fisher

$$\rho(\phi',\phi) = \frac{1}{Z}$$

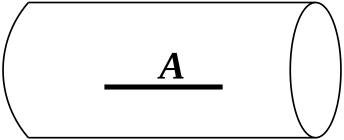


$$Z =$$



Reduced density matrix

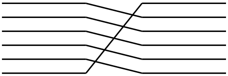
$$\rho_A = \frac{1}{Z}$$



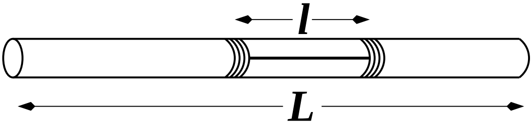
- – sew together the edges along $\tau = 0, \beta$, leaving slit(s) open along A
- now use ‘replica trick’

$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A = -\left.\frac{\partial}{\partial n}\right|_{n=1} \mathrm{Tr}\,\rho_A^n$$

- for integer $n \geq 1$, $\mathrm{Tr}\rho_A^n$ (the Renyi/Tsallis entropy) is given by n copies of the path integral for S_A, sewn together cyclically along the slits: an n-sheeted Riemann surface



High-temperature limit $\beta \to 0$



$$Z_n \approx Z_1(\ell, n\beta)\, Z_1(L-\ell, \beta)^n$$

$$\mathrm{Tr}\,\rho_A^n \approx \frac{Z_1(\ell, n\beta)}{Z_1(\ell,\beta)^n} \sim \frac{\exp\big(-n\beta\, F_A(n\beta)\big)}{\exp\big(-n\beta\, F_A(\beta)\big)}$$

- differentiating wrt n at $n=1$

$$S_A \sim \beta(E_A - F_A)$$

- so in this limit S_A is the thermodynamic entropy of A

The critical case in $d = 1$

- suppose the 1d system is at a quantum critical point with $z = 1$ (a linear dispersion relation $\omega = v|k|$)
- dimensional analysis (Stefan's law):

$$F(\beta) = -\pi c\ell/6\beta^2$$

in units where $\hbar = v = 1$ and central charge $c = 1$ for a single species of boson

- so for $\ell \gg \beta$

$$\mathrm{Tr}\,\rho_A^n \sim \exp\left[-\frac{\pi c}{6\beta}\left(n - \frac{1}{n}\right)\ell\right]$$

- for $d = 1$, $z = 1$ this is related to the opposite limit $\beta \to \infty$ by conformal symmetry
- the conformal map $z \to (\beta/2\pi)\log z$ converts exponential decay along cylinder into power-law decay at $T = 0$:

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so (Holzhey *et al* 1994)

$$S_A \sim (c/3)\log \ell$$

- in $d = 1$, S grows only logarithmically, even at a critical point: responsible for success of DMRG
- logarithmic behaviour not restricted to $z = 1$ critical points but also eg in random quantum spin chains (Refael and Moore)

- many more universal results, eg finite-temperature cross-over between entanglement and thermodynamic entropy (Korepin 2004, Calabrese + JC 2004):

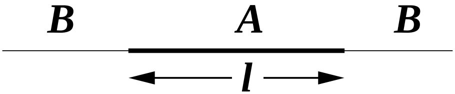
$$S_A \sim (c/3)\log\big((\beta/\pi)\sinh(\pi\ell/\beta)\big)$$

- in general, if A and B each consist of several disconnected pieces,

$$S_A \sim \tfrac{1}{3}\int_{x\in A}\int_{y\in B}(x-y)^2\langle T(x)T(y)\rangle\,dx\,dy$$

where T is the energy-momentum tensor

Finite correlation length in $d = 1$



- entanglement occurs only over a distance $\sim \xi$ of the contact points between A and B

$$S_\ell \sim 2 \times (c/6) \log \xi \qquad (1)$$

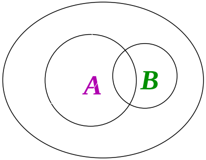
with universal corrections $O(e^{-2\ell/\xi})$ ((JC,Castro-Alvaredo,Doyon 2007))

- (1) also follows in lattice models from the corner transfer matrix ((Peschel 2004, Calabrese + JC 2004))

Higher dimensions $d > 1$

- in general the leading term in S_A is nonuniversal $\sim a^{1-d}\,\mathrm{Area}(\partial A)$: the area law
- expect a universal term $\sim \xi^{1-d} \times$ area hidden behind this
- however these ‘area’ terms cancel in

$$S_{A\cup B} + S_{A\cap B} - S_A - S_B$$



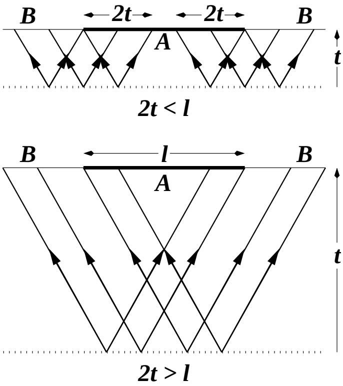
- for $z = 1$ quantum critical points this quantity is expected to be universal and given by $c \times$ geometrical factor where c depends on the universality class

B. Time-dependence after a quantum quench

- prepare system at time $t = 0$ in pure state $|\Psi_0\rangle$ which is the ground state of some hamiltonian H_0 with a gap m_0
- for times $t > 0$ evolve *unitarily* with hamiltonian H with $[H, H_0] \neq 0$ – no dissipation, no noise
- relevant to experiments on cold atoms in optical lattices
- how do the reduced density matrix ρ_A, its entropy S_A and correlation functions of local operators $\mathcal{O}(x)$ with $x \in A$ evolve?
- does ρ_A reach a stationary state (and if so what?)

- these questions have been answered in some solvable cases (Barouch and McCoy 1970; Sengupta *et al* 2004; Calabrese + JC 2005*ff*; Peschel 2007; Sotiriadis + JC):
- when H is quadratic in a, $a^\dagger$ (free bosons or fermions)
- when H corresponds to a conformal field theory
- in interacting field theory in a self-consistent (Hartree) approximation
- they all lead to the following simple physical predictions:

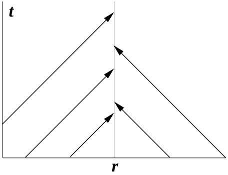
1. Horizon effect



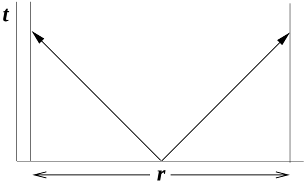
- $|\Psi_0\rangle$ acts as source for quasiparticles
- left and right moving particles originating from points within $\sim \xi_0$ are entangled
- $S_A \propto$ number of particles arriving on A entangled with those arriving on B. This increases $\propto t$ up to $t \sim \ell/2v$ and then saturates at a value $\propto \ell$

Correlation functions

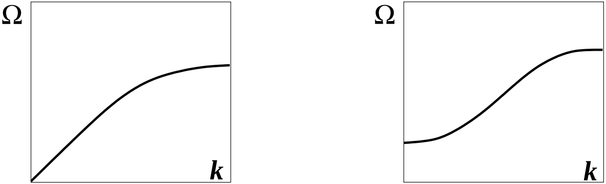
- 1-point functions $\langle \mathcal{O}(x,t) \rangle$ typically relax exponentially due to incoherent radiation



- connected 2-point functions $\langle \mathcal{O}(x_1,t)\mathcal{O}(x_2,t) \rangle_c$ do not change until x_1, x_2 fall into the horizon, after which the full correlation function saturates



General dispersion relation



- for more general quasiparticle dispersion relation $\omega(k)$ horizon effect depends on $v_{\text{group}} = d\omega_k/dk$: approach to stationary behaviour controlled by slowest quasiparticles

2. Thermalisation at late times

- consider first a simple harmonic oscillator, quenched from frequency $\omega_0 \to \omega$ with $\omega_0 \gg \omega$:
- overlap between the ground state $|\psi_0\rangle$ of H_0 and an excited state $|E\rangle$ of H is

  $$\langle \psi_0 | E \rangle \propto \exp(-\beta_{\rm eff} E/2)$$

  where $\beta_{\rm eff} \sim 4/\omega_0$
- density matrix elements

  $$\langle E | \rho(t) | E' \rangle \sim e^{-\beta_{\rm eff}(E+E')/2}\, e^{i(E-E')t/\hbar}$$

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- a single harmonic oscillator does not reach a thermal steady state – it oscillates with frequency ω!

. . . however, in an extensive system. . .

- if we look in a finite region A of size $\ll t/v$ as $t \to \infty$, the different k-modes have frequencies $\omega_k = m + O(k^2)$ and destructively interfere except near $k = 0$
- ρ_A and correlation functions of local observables $\mathcal{O}(x)$ with $x \in A$ become stationary as $t \to \infty$, as though they were at finite effective temperature
- a more detailed analysis gives an effective inverse temperature

$$\beta_{\text{eff}} = (4/m)\tanh^{-1}\left(m/m_0\right)$$

- similarly the extensive part of S_A saturates at a value equal to the Gibbs-Boltzmann entropy corresponding to this effective temperature
- this is despite the fact the the whole system remains in a pure state and there is no ergodicity or coupling to a heat bath – the effect arises as a consequence of destructive quantum interference

Summary

- entanglement entropy provides a useful order-parameter independent diagnostic of quantum phase transitions, with many universal features
- after a quantum quench, the entropy and other local quantities should reach a stationary state corresponding to finite effective temperature
- many open questions:
  - how robust is this behaviour?
  - what about quenches into an ordered phase?
  - . . .

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- after a quantum quench, the entropy and other local quantities should reach a stationary state corresponding to finite effective temperature
- many open questions:
    - how robust is this behaviour?
    - what about quenches into an ordered phase?
    - . . .
- maybe answered at STATPHYS 24?