EFFECT OF QUENCHED IMPURITIES
on
FIRST-ORDER TRANSITIONS

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Outline.

• random fields vs. ‘random bonds’
• Imry-Ma-Imbrie argument for random fields
• analog argument for random bonds (Aizenman-Wehr)
• what happens beyond the crossover length?
• numerical work on $Q$-state Potts model
• a more-or-less solvable example: $N$ coupled Ising models
• experiment

These transparencies available at
http://www-thphys.physics.ox.ac.uk/users/JohnCardy/seminars/rfo.ps
The Problem

• pure system with a unique high-$T$ phase and $Q \geq 2$ equivalent low-$T$ phases;

• transition at $T_c$ in pure system is first-order:
  • coexistence at $T_c$ of $Q + 1$ phases with same free energy at $T_c$
  • discontinuities in internal energy $U$ (latent heat $L$) and order parameter $M$
  • finite correlation length $\xi_0$ at $T_c$
  • finite interfacial tensions at $T_c$

• add quenched uncorrelated random impurities, either:
  • random fields, coupling to local order parameter $M(r)$
    \[
    \mathcal{H} = \mathcal{H}_{\text{pure}} + \sum_r h(r)M(r); \quad h(r) = \pm h
    \]
  • or ‘random $T_c$’, coupling only to local energy density $U(r)$
    \[
    \mathcal{H} = \mathcal{H}_{\text{pure}} + \sum_r \Delta t_c(r)U(r); \quad \Delta t_c(r) = \pm \Delta
    \]

• Random field Ising model and random $T_c$ are physically very different but mathematically analogous
Random Field Ising Model – Imry-Ma argument

- look at low temperature (ignore thermal fluctuations)
- consider stability of ordered phase against formation of region of oppositely ordered phase, size $O(R)$

![Diagram showing arrows and region labeled R]

- by choosing region appropriately, reduce energy by $\sim hM_0 \sqrt{R^d}$
- increase of energy due to domain wall $\sim JR^{d-1}$
- $\Rightarrow$ for $d > 2$, ordered phase is stable at low $T$ against weak random fields
• $d = 2$ more subtle: in fact ordered phase is unstable [Imbrie]

• RG argument [Bray-Moore] based on looking at a single interface gives

\[
\begin{align*}
\frac{dh}{d\ell} &= (d/2)h \\
\frac{dJ}{d\ell} &= J((d - 1) - (h/J)^2 + \cdots)
\end{align*}
\]

so that

\[
\frac{d(h/J)}{d\ell} = (1 - d/2)(h/J) + (h/J)^3 + \cdots
\]

− critical value $h^* = O(J)$ for $d > 2$

− finite correlation length $\xi \sim e^{\text{const}(J/h)^2}$ for $d = 2$
Random $T_c$ Analogy

magnetization $M \leftrightarrow$ internal energy $U$
random field $h \leftrightarrow$ random $T_c \sim \Delta$
ignore thermal fluctuations $\leftrightarrow$ strong first-order
interfacial tension $J \leftrightarrow$ interfacial tension $L\xi_0$

Then:

- for $d > 2$ there is critical $\Delta^* = O(L\xi_0)$ above which phase coexistence cannot occur
- thermal exponents at this tricritical point are related to the magnetic exponents of the random field Ising model [Jacobsen & Cardy]
- for $d = 2$ any amount of randomness destroys phase coexistence [Aizenman-Wehr] and there is a cross-over length scale $\xi_\times \sim \xi_0 e^{\text{const}(L\xi_0/\Delta)^2}$

BUT:

- what is the nature of the critical state for $\Delta > \Delta^*$ for $d > 2$, and on length scales $\gg \xi_\times$ in $d = 2$?
- why is the random $T_c$ case different from the random field Ising model on length scales $\gg \xi_\times$?
System breaks up into domains size $O(\xi_x)$:

For $Q \geq 2$ the ordered regions can still fluctuate - but what is the resultant critical behaviour?

- does the correlation length diverge?
- what are the critical exponents?
Numerical Results for the $Q$-state Potts Model

Pure system:

- ‘spins’ $s(r)$ taking values 1, 2, ..., $Q$
- $\mathcal{H} = -J \Sigma_{\text{n.n.}} \delta_{s(r),s(r')}$
- $Q$ ordered states for $T < T_c$
- first-order for $Q > Q_c = 4$ in $d = 2$
- strongly first-order for large $Q$: $L \sim \ln Q$, $\xi_0 = O(1)$
- $T_c$ known exactly by duality for $d = 2$

Now add randomness to $J$. Results ($d = 2$):

- correlation length appears to diverge
- $\nu$ numerically consistent with 1
- $\beta$ increases steadily with $Q$, asymptoting to $\beta \approx 0.19$
- effective central charge $c \approx \frac{1}{2} \ln_2 Q$

Comment: $c$ is a measure of the entropy of the fluctuating large-scale degrees of freedom $\Rightarrow c \propto \ln Q$. But why the coefficient?
A Solvable Case: $N$ coupled Ising models
JC (1995 & unpublished)

Pure system:

- Ising spins $s_j(r) = \pm 1$ ($j = 1, \ldots, N$)

\[ H = -J \sum_j \sum_{\text{n.n.}} s_j(r)s_j(r') - g \sum_{jk} \sum_{\text{n.n.}} ((s_j(r)s_j(r'))(s_k(r)s_k(r'))) \]

- for $g > 0$ model undergoes a first-order transition if $N > 2$ (see below)

Now add quenched randomness in $J$: RG flows
• At large enough length scales, $N$ strongly coupled Ising models with weak disorder
  $\approx N$ weakly coupled Ising models with strong disorder
• if so, and such transitions are ‘superuniversal’, this explains
  – why $\nu \approx 1$
  – why $c \approx \frac{1}{2} \ln_2 Q = \frac{1}{2} N$ where $2^N = Q$
• but $\beta$ is apparently not superuniversal
Light from the fermion description of the 2d Ising model

- continuum limit Ising model $\leftrightarrow$ free fermion
  $\mathcal{H} = i(\bar{\psi}\sigma \cdot \partial \psi + m\bar{\psi}\psi)d^2r$ where $m \propto T - T_c$

- $N$ coupled Ising models
  
  $Z = \int [d\psi] e^{-\int \Sigma_j(\bar{\psi}_j \sigma \cdot \partial \psi_j + m\bar{\psi}_j \psi_j) - g(\Sigma_j \bar{\psi}_j \psi_j)^2}d^2r$
  $= \int [d\psi][du] e^{-\int (u^2/g + \Sigma_j(\bar{\psi}_j \sigma \cdot \partial \psi_j + (m+u)\bar{\psi}_j \psi_j))d^2r}$
  $= \int [du] e^{-\int (u^2/g)d^2r} (e^{-F_{\text{Ising}}(m+u)})^N$

- rescale $g \rightarrow g/N$: large $N$ saddle-point found by minimizing
  $F/N = F_{\text{Ising}}(m+u) + u^2/g \sim (m+u)^2 \ln |m+u| + u^2/g$

\begin{itemize}
  \item for $m \rightarrow 0^+$, $u \sim u_0 \sim e^{-1/g}$
  \item for $m \rightarrow 0^-$, $u \sim -u_0 \Rightarrow$ first-order transition!
  \item now add randomness $m \rightarrow m(r)$:
    \begin{align*}
      m(r) \text{ predominantly } > 0 & \Rightarrow u \sim u_0 \\
      m(r) \text{ predominantly } < 0 & \Rightarrow u \sim -u_0
    \end{align*}
\end{itemize}

- the Imry-Ma domains, smoothed out on scale $\xi_0$
• but along lines where $u = 0$ there are gapless fermions!

• $N$ decoupled fermions ($= N$ decoupled Ising models) in disorder correlated over regions $\sim \xi_x$
• picture much clearer in network model description of random Ising models [Chalker and Merz, 2001]

• surviving fermions completely analogous to edge states in quantum Hall physics!
Summary & Open Questions

• strong analogy between random field Ising model and random $T_c$ at a first-order transition $\Rightarrow$ Imry-Ma domains

• for $d = 2$, one model for which nature of large scale critical state understood, analogy to quantum Hall physics

• superuniversality of energy fluctuations: $\nu \sim 1$, $c \sim \frac{1}{2} \ln_2 Q$

• why are order parameter fluctuations not superuniversal?

• experimental realizations:
  - difficult to realize random $T_c$ without random fields
  - very large length scales necessary, need good samples to get sharp transition