QUANTUM AND CLASSICAL NETWORK MODELS

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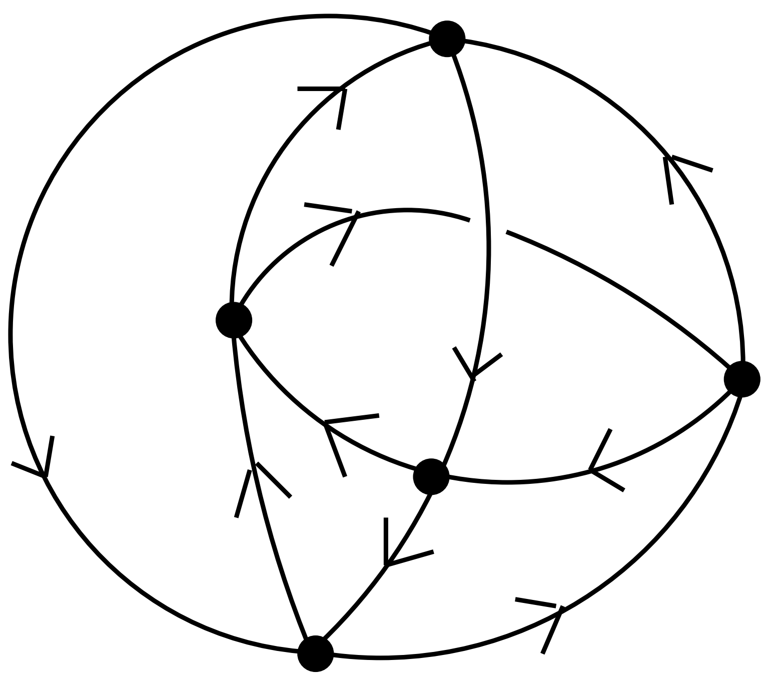
Earlier work by Gruzberg,Ludwig,Read;
Beamond,JC,Chalker

This talk is about some mathematical results on particle transport in random media

- propagation of a particle in a random medium is often modeled by a *network*:
- medium represented by some fixed graph $\mathcal{G}$ with $\mathcal{E}$ edges and $\mathcal{N}$ nodes
- at each tick of the clock, the particle moves through one node, from edge e to edge e' say



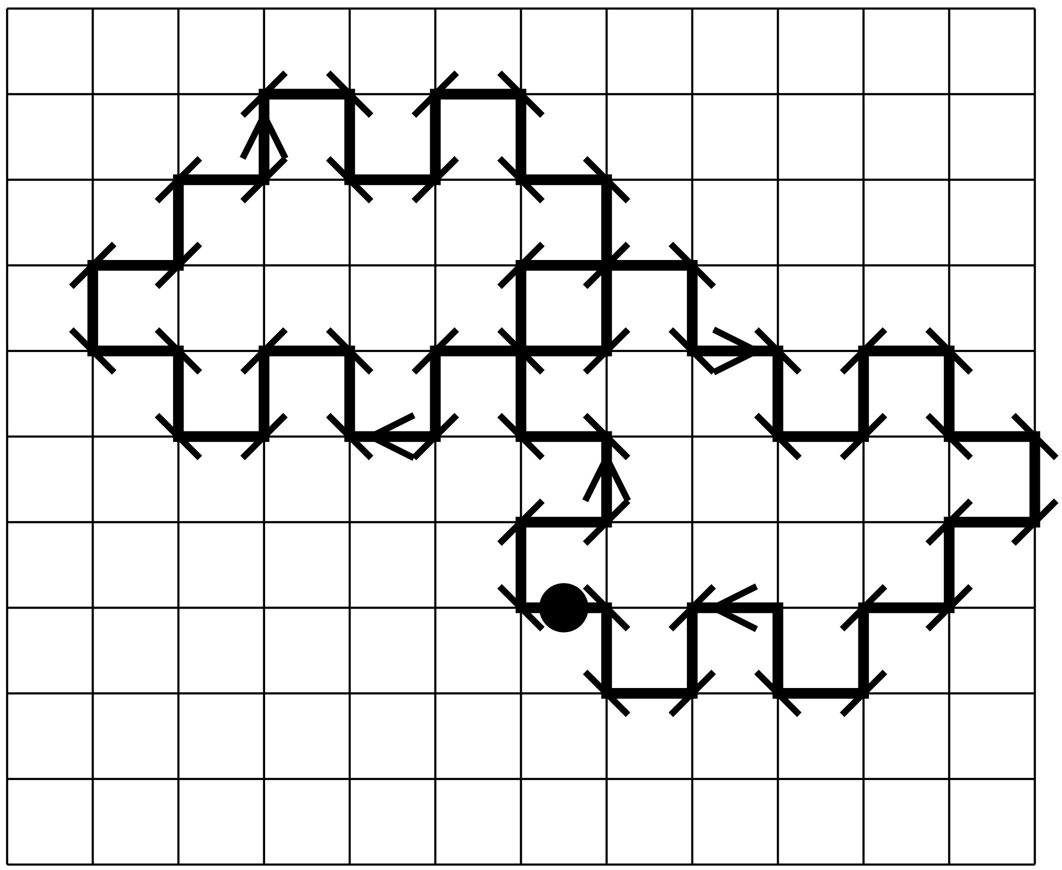
Classical Version

- specify *transition probabilities* $\Omega_n(e \rightarrow e')$ at each node n, with

$$\Sigma_{e'}\,\Omega_n(e \rightarrow e') = 1$$

- gives a random walk on $\mathcal{G}$:
  - what is the probability $P(e_1, e_2; T)$ that, beginning at e_1, particle reaches e_2 after T steps?
- more interesting if the walk is history-dependent, eg:
  - a given edge may be traversed at most once: a *trail*
  - transition probabilities $\Omega_n(e \rightarrow e')$ may depend on whether particle has previous visited the node

Example: Hall of Mirrors



- at each node the particle turns R or L with probabilities p and $1 - p$ respectively
- if it visits node again, it must turn the same way
- can either fix the orientation of the mirrors in advance, or fix them once particle has passed through node

$\Rightarrow$ is the particle localized, or can it escape to infinity? $\Leftarrow$

- for this particular model, escape possible only if $p = \frac{1}{2}$

Quantum Version

- Hilbert space $\mathcal{H}_e$ of states on each edge e

$$\mathcal{H} = \oplus_e \mathcal{H}_e$$

- propagation through a node n described by an S-matrix:

$$S_n(e', e) : \mathcal{H}_e \to \mathcal{H}_{e'}$$

with

$$\Sigma_{e'} S_n(e', e)^\dagger S_n(e', e) = 1$$

- propagation along an edge e described by

$$U_e : \mathcal{H}_e \to \mathcal{H}_e$$

- full evolution operator $\mathcal{U}$ is a direct sum over edges and nodes of $\mathbf{1} \otimes \ldots \otimes U_e \otimes \ldots \otimes \mathbf{1}$ and $\mathbf{1} \otimes \ldots \otimes S_n \otimes \ldots \otimes \mathbf{1}$.

Green's function

$$G(e_2, e_1; z) \equiv \langle e_2 | (1 - z\mathcal{U})^{-1} | e_1 \rangle$$

- G may be expanded in powers of z as a sum of Feynman paths on $\mathcal{G}$ from e_1 to e_2, each weighted by a product of U_e for each edge and elements of S_n for each node
  - a given edge may be traversed more than once
- if $\mathcal{G}$ is *closed* , eigenvalues ϵ_j of $\mathcal{U}$ are on unit circle: density of states

$$\rho(\epsilon) = \Sigma_j \delta(\epsilon - \epsilon_j) =$$
$$\frac{1}{4\pi\mathcal{E}} \Sigma_e \lim_{z \to e^{i\epsilon}} \mathrm{Tr}\left(G(e, e; z) - G(e, e; 1/z^*)\right)$$

Transport properties

Consider an *open* graph $\mathcal{G}$ with open edges $\{e_{\rm in}\}$ and $\{e_{\rm out}\}$.
Transmission matrix

$$t = \langle e_{\rm out}|(1-\mathcal{U})^{-1}|e_{\rm in}\rangle$$

Landauer formula for conductance

$$g = \mathrm{Tr}\, t^\dagger t = \mathrm{Tr}\, G(e_{\rm out}, e_{\rm in}; 1)^\dagger G(e_{\rm out}, e_{\rm in}; 1)$$

- interference effects between different Feynman paths make this problem hard
- if all the U_e and S_n are independent random variables in general this leads to Anderson localization: e.g. on an infinite planar graph all states are localized: $g \to 0$ as $|e_{\rm out} - e_{\rm in}| \to \infty$.
- however this can be evaded if the single-particle hamiltonian H satisfies special symmetries

Examples

1. particle in a strong magnetic field (quantum Hall effect)

$$H = -i\vec{A} \cdot \vec{\nabla}_x + V(x)$$

so $H_V^* = -H_{-V}$: if distribution of V is symmetric about 0 then states $\pm E$ are paired: $E = 0$ is special and can be delocalized even in two dimensions

- corresponding network model (Chalker-Coddington model) is on a fully directed graph $\mathcal{G}$, with $U_e \in \mathrm{U}(1)$ being quenched random variables: this model is unsolved for any interesting graphs

2. (subject of this talk)
– suppose there exists a symmetry (class C)

$$\sigma_y \, H \, \sigma_y = -H^*$$

Corresponding class C network model:

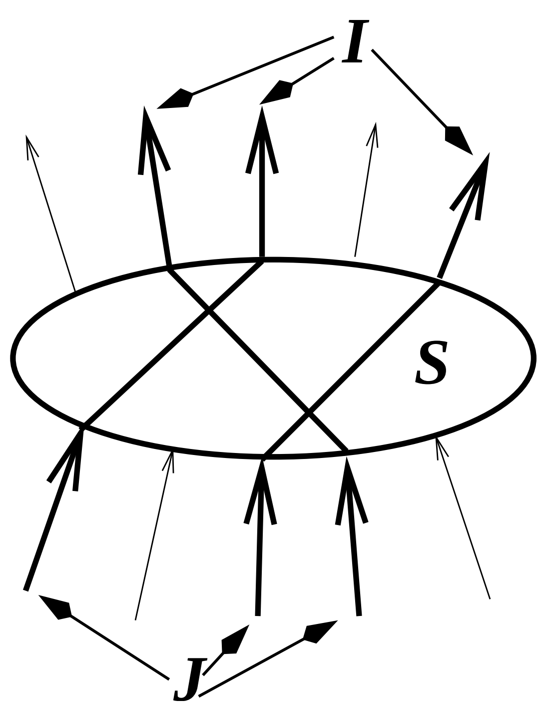
- $\sigma_y \mathcal{U} \, \sigma_y = \mathcal{U}^*$
- take each $\mathcal{H}_e$ to be 2-dimensional (e.g. particle is an electron with spin)
- $U_e \in \mathrm{SU}(2)$, quenched random variables with Haar measure on $\mathrm{SU}(2)$
- wlog can choose S_n diagonal in $\mathrm{SU}(2)$ indices, so $S_n \in \mathrm{O}(N)$ for a node with N incoming and N outgoing edges

Theorem 1. *The mean of $G(e_1, e_2; z)$ vanishes unless $e_1 = e_2$, in which case it is given by*

$$\mathrm{Tr}\,\overline{G(e,e;z)} = \begin{cases} 2 - \Sigma_{\tau(e)} w_{\tau(e)}\, z^{2|\tau(e)|} & |z| < 1 \\ \Sigma_{\tau(e)} w_{\tau(e)}\, z^{-2|\tau(e)|} & |z| > 1 \end{cases}$$

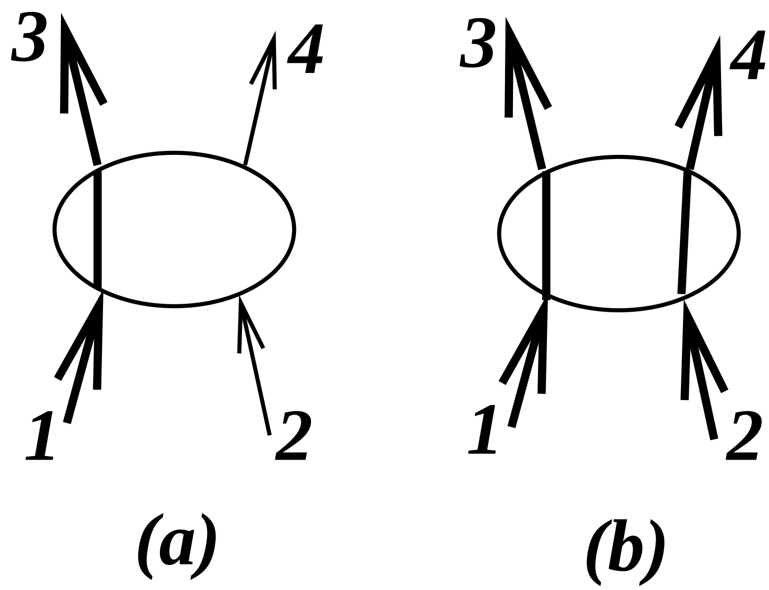
where the sums are over all closed trails $\tau(e)$ rooted at e and $w_{\tau(e)}$ is given by the product over all the nodes on $\tau(e)$ of $\Omega(I_{n;\tau}; J_{n;\tau})$

$$\equiv (-1)^{\pi_{n;\tau}} \prod_{j \in J_{n;\tau}} S_{\pi_{n;\tau}(j),j} \quad (\det S_{I_{n;\tau}, J_{n;\tau}})$$



Remark. $\Sigma_{I_n}\Omega(I_n; J_n) = 1$

Example



(a) gives $\quad S_{31}\,(S_{31}) = S_{31}^2$

(b) gives $\quad S_{31}S_{42}\,(S_{31}S_{42} - S_{32}S_{41}) = S_{31}^2$

Theorem 2. *The mean point conductance $\overline{g}$ between e_{in} and e_{out} is the sum of over all open trails connecting the two edges, weighted as in Theorem 1.*

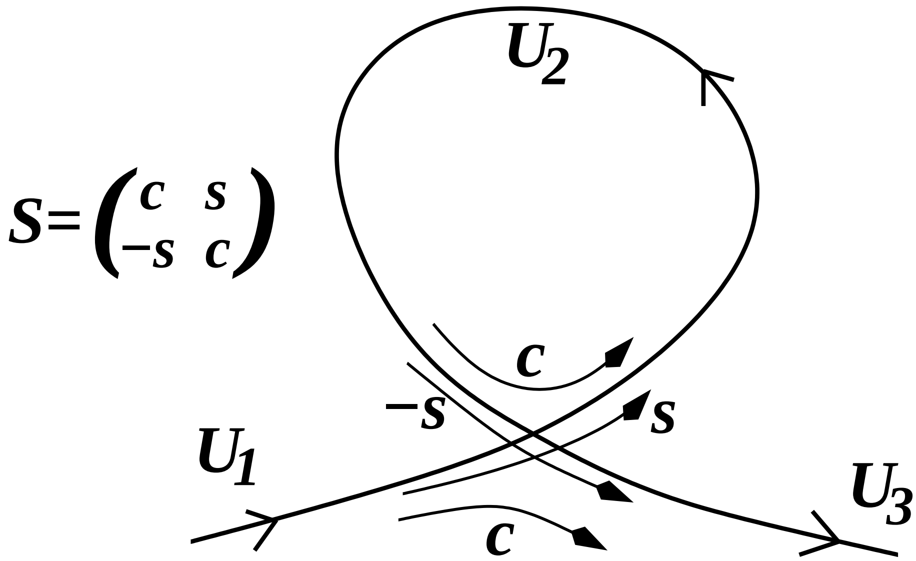
Mean conductance is equal to the probability that a trail starting at e_{in} reaches e_{out}.

Proof uses special properties of SU(2) matrices:

- group manifold is S_3: any such matrix may be parametrized as
$$U = e^{i\alpha\,\sigma\cdot\mathbf{n}} = \cos\alpha\,\mathbf{1} + i\sin\alpha\,\sigma\cdot\mathbf{n}$$
where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\mathbf{n} \in S_2$
- any real linear combination of SU(2) matrices is equal to a real number times an SU(2) matrix
- Haar measure is uniform measure on $S_3 \propto \cos^2\alpha\, d\alpha d\Omega_{\mathbf{n}} = \frac{1}{2}(1-\cos 2\alpha) d\alpha d\Omega_{\mathbf{n}}$
- so
$$\overline{U^p} = \begin{cases} 1 & : \quad p = 0 \\ -\frac{1}{2} & : \quad p = 2 \\ 0 & : \quad \text{otherwise} \end{cases}$$

Therefore if $G = G(e_{\text{out}}, e_{\text{in}}; 1)$,
$$\overline{G^\dagger G} = -2\overline{G^2} = \overline{(\det G)^2}\,\mathbf{1}$$

Simple Example



$$G \;=\; c\,U_3U_1 + z(-s)\,s\,U_3U_2U_1$$
$$+z^2((-s)\,c\,s\,U_3U_2U_2U_1 + \cdots$$

$$G^2 = c^2\,(U_3U_1)^2 + z^2\,s^4\,(U_3U_2U_1)^2$$
$$-z^2\,s^2\,c^2\,(U_3U_1U_3U_2U_2U_1 + U_3U_2U_2U_1U_3U_1) + \cdots$$

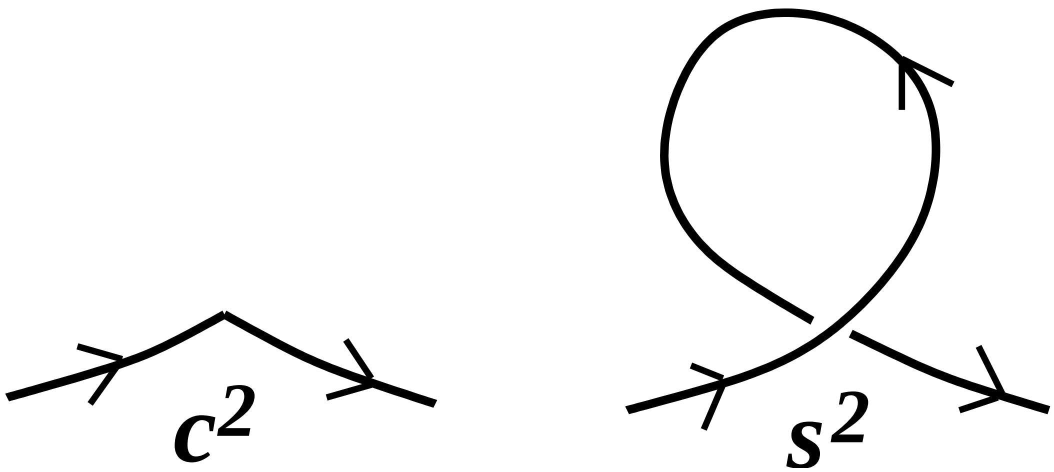
Write $U_3U_1U_3U_2U_2U_1 = (U_3U_1)^2(U_1^{-1}U_2U_1)^2$

$$\overline{G^2} = (-\tfrac{1}{2})c^2 + z^2(-\tfrac{1}{2})s^4 - 2(-\tfrac{1}{2})^2 s^2 c^2$$

so

$$\overline{G^\dagger G} = \cos^2\theta + z^2\sin^2\theta$$

Equivalent classical problem



Probability that trail from e_{in} ends at e_{out}

$$= c^2 + z^2 s^2$$

($= 1$ if $z = 1$).

Proof for a general graph uses **supersymmetry**

- write $(1 - z\mathcal{U})^{-1}$ as a gaussian integral over variables $b_R(e)$ and $b_L(e)$ localized at the ends of each edge:

$$G = \frac{\int \Pi_e [db_L(e)][db_R(e)] b_L(e_2) b_L^\dagger(e_1)\, e^{W_b}}{\int \Pi_e [db_L(e)][db_R(e)]\, e^{W_b}}$$

where $W_b = W_{\text{edge}} + W_{\text{node}}$ with

$$\begin{aligned} W_{\text{edge}} &= z\Sigma_e\, b_L^\dagger(e) U_e b_R(e) \\ W_{\text{node}} &= \Sigma_n \Sigma_a \Sigma_{ij}\, b_{Ra}^*(e_i)(S_n)_{ij} b_{La}(e_j) \end{aligned}$$

and the integration is wrt the coherent state measure

$$\int [db] = (1/\pi^2) \int e^{-b^\dagger b}\, \Pi_a d\mathrm{Re}\, b_a\, d\mathrm{Im}\, b_a$$

- however in this form it is difficult to average over the U_e

- introduce corresponding anticommuting Grassmann variables $f_{R,L}(e)$, $\bar{f}_{R,L}(e)$

with

$$\int [df] = \int d\bar{f} df \, e^{-\bar{f} f}$$

so that

$$\int [df] f = \int [df] \bar{f} \; = \; 0;$$
$$\int [df] \, 1 = \int [df] f \bar{f} \; = \; 1$$

Then

$$G(e_2, e_1) \; = \; \int \Pi_e [db_L(e)][db_R(e)][df_L(e)][df_R(e)] \\ b_L(e_2) b_L^\dagger(e_1) \, e^{W_b + W_f}$$

- $W_b + W_f$ is invariant under global supersymmetry rotations.

Quenched average over the U_e

On each edge we have

$$\int dU\ \exp(z b_L^\dagger U b_R + z \bar{f}_L U f_R)$$

Lemma: *this equals*

$$\begin{aligned}
&1 + \\
&+\tfrac{1}{2} z^2 (b_{L1}^* \bar{f}_{L2} - b_{L2}^* \bar{f}_{L1})(b_{R1} f_{R2} - b_{R2} f_{R1}) \\
&+z^2 (\bar{f}_{L1} \bar{f}_{L2})(f_{R2} f_{R1})
\end{aligned}$$

- interpretation: after averaging, all that can propagate along a given edge is:
  - 1, or
  - a fermion-boson pair $(1/\sqrt{2})(b_1 f_2 - f_1 b_2)$
  - a fermion-fermion pair $f_1 f_2$

- to compute $\overline{G}^2$, we can follow the propagation of (e.g.) a fermion-fermion pair
- it follows a *trail* through $\mathcal{G}$
- at a given node, we contract the incoming $\bar{f}_1\bar{f}_2$ pairs on edges J onto the outgoing f_1f_2 pairs on edges I using Wick's theorem
- this gives rise to the factors $\Omega(I;J)$ in Theorem 1.