QUANTUM AND CLASSICAL NETWORK MODELS

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Earlier work by Gruzberg, Ludwig, Read; Beamond, JC, Chalker
This talk is about some mathematical results on particle transport in random media

- propagation of a particle in a random medium is often modeled by a network:
- medium represented by some fixed graph $G$ with $E$ edges and $N$ nodes
- at each tick of the clock, the particle moves through one node, from edge $e$ to edge $e'$ say
Classical Version

• specify transition probabilities $\Omega_n(e \rightarrow e')$ at each node $n$, with

\[ \sum_{e'} \Omega_n(e \rightarrow e') = 1 \]

• gives a random walk on $\mathcal{G}$:

  – what is the probability $P(e_1, e_2; T)$ that, beginning at $e_1$, particle reaches $e_2$ after $T$ steps?

• more interesting if the walk is history-dependent, eg:

  – a given edge may be traversed at most once: a trail

  – transition probabilities $\Omega_n(e \rightarrow e')$ may depend on whether particle has previous visited the node
Example: Hall of Mirrors

- at each node the particle turns R or L with probabilities $p$ and $1 - p$ respectively
- if it visits node again, it must turn the same way
- can either fix the orientation of the mirrors in advance, or fix them once particle has passed through node
⇒ is the particle localized, or can it escape to infinity? ⇐

• for this particular model, escape possible only if $p = \frac{1}{2}$
Quantum Version

• Hilbert space $\mathcal{H}_e$ of states on each edge $e$

$$\mathcal{H} = \bigoplus_e \mathcal{H}_e$$

• propagation through a node $n$ described by an $S$-matrix:

$$S_n(e', e) : \mathcal{H}_e \to \mathcal{H}_{e'}$$

with

$$\sum_{e'} S_n(e', e)^\dagger S_n(e', e) = 1$$

• propagation along an edge $e$ described by

$$U_e : \mathcal{H}_e \to \mathcal{H}_e$$

• full evolution operator $\mathcal{U}$ is a direct sum over edges and nodes of $\mathbf{1} \otimes \ldots \otimes U_e \otimes \ldots \otimes \mathbf{1}$ and $\mathbf{1} \otimes \ldots \otimes S_n \otimes \ldots \otimes \mathbf{1}$. 
Green’s function

\[ G(e_2, e_1; z) \equiv \langle e_2 | (1 - zU)^{-1} | e_1 \rangle \]

- \( G \) may be expanded in powers of \( z \) as a sum of Feynman paths on \( \mathcal{G} \) from \( e_1 \) to \( e_2 \), each weighted by a product of \( U_e \) for each edge and elements of \( S_n \) for each node
  - a given edge may be traversed more than once

- if \( \mathcal{G} \) is \textit{closed} , eigenvalues \( \epsilon_j \) of \( U \) are on unit circle: density of states

\[
\rho(\epsilon) = \sum_j \delta(\epsilon - \epsilon_j) = \\
\frac{1}{4\pi\mathcal{E}} \sum_e \lim_{z \to e^{i\epsilon}} \text{Tr} \left( G(e, e; z) - G(e, e; 1/z^*) \right)
\]
Transport properties

Consider an open graph $\mathcal{G}$ with open edges $\{e_{\text{in}}\}$ and $\{e_{\text{out}}\}$.

Transmission matrix

$$t = \langle e_{\text{out}} | (1 - U)^{-1} | e_{\text{in}} \rangle$$

Landauer formula for conductance

$$g = \text{Tr} \ t^\dagger t = \text{Tr} \ G(e_{\text{out}}, e_{\text{in}}; 1)^\dagger G(e_{\text{out}}, e_{\text{in}}; 1)$$

- interference effects between different Feynman paths make this problem hard
- if all the $U_e$ and $S_n$ are independent random variables in general this leads to Anderson localization: e.g. on an infinite planar graph all states are localized: $g \to 0$ as $|e_{\text{out}} - e_{\text{in}}| \to \infty$.
- however this can be evaded if the single-particle hamiltonian $H$ satisfies special symmetries
Examples

1. particle in a strong magnetic field (quantum Hall effect)

\[ H = -i \vec{A} \cdot \vec{\nabla}_x + V(x) \]

so \( H^*_V = -H_{-V} \): if distribution of \( V \) is symmetric about 0 then states \( \pm E \) are paired: \( E = 0 \) is special and can be delocalized even in two dimensions

- corresponding network model (Chalker-Coddington model) is on a fully directed graph \( \mathcal{G} \), with \( U_e \in U(1) \) being quenched random variables: this model is unsolved for any interesting graphs
2. (subject of this talk)
- suppose there exists a symmetry (class C)

\[ \sigma_y H \sigma_y = -H^* \]

Corresponding class C network model:

- \[ \sigma_y \mathcal{U} \sigma_y = \mathcal{U}^* \]
- take each \( \mathcal{H}_e \) to be 2-dimensional (e.g. particle is an electron with spin)
- \( U_e \in SU(2) \), quenched random variables with Haar measure on \( SU(2) \)
- wlog can choose \( S_n \) diagonal in \( SU(2) \) indices, so \( S_n \in O(N) \) for a node with \( N \) incoming and \( N \) outgoing edges
Theorem 1. The mean of $G(e_1, e_2; z)$ vanishes unless $e_1 = e_2$, in which case it is given by

$$\text{Tr} \, G(e, e; z) = \begin{cases} 2 - \sum_{\tau(e)} w_{\tau(e)} z^{2|\tau(e)|} & |z| < 1 \\ \sum_{\tau(e)} w_{\tau(e)} z^{-2|\tau(e)|} & |z| > 1 \end{cases}$$

where the sums are over all closed trails $\tau(e)$ rooted at $e$ and $w_{\tau(e)}$ is given by the product over all the nodes on $\tau(e)$ of $\Omega(I_{n;\tau}; J_{n;\tau})$

$$\equiv (-1)^{\pi_{n;\tau}} \prod_{j \in J_{n;\tau}} S_{\pi_{n;\tau}(j), j} \quad (\det S_{I_{n;\tau}, J_{n;\tau}})$$
Remark. $\sum I_n \Omega(I_n; J_n) = 1$

Example

(a)

(b)

(a) gives $S_{31} (S_{31}) = S_{31}^2$

(b) gives $S_{31} S_{42} (S_{31} S_{42} - S_{32} S_{41}) = S_{31}^2$
Theorem 2. The mean point conductance $\bar{g}$ between $e_{in}$ and $e_{out}$ is the sum of over all open trails connecting the two edges, weighted as in Theorem 1.

Mean conductance is equal to the probability that a trail starting at $e_{in}$ reaches $e_{out}$. 
Proof uses special properties of SU(2) matrices:

- group manifold is $S_3$: any such matrix may be parametrized as

$$U = e^{i\alpha \sigma \cdot \mathbf{n}} = \cos \alpha \mathbf{1} + i \sin \alpha \sigma \cdot \mathbf{n}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\mathbf{n} \in S_2$

- any real linear combination of SU(2) matrices is equal to a real number times an SU(2) matrix

- Haar measure is uniform measure on $S_3 \propto \cos^2 \alpha \ d\alpha d\Omega_\mathbf{n} = \frac{1}{2} (1 - \cos 2\alpha) d\alpha d\Omega_\mathbf{n}$

- so

$$\overline{U^p} = \begin{cases} 1 & : p = 0 \\ -\frac{1}{2} & : p = 2 \\ 0 & : \text{otherwise} \end{cases}$$

Therefore if $G = G(e_{\text{out}}, e_{\text{in}}; 1)$,

$$\overline{G^\dagger G} = -2\overline{G^2} = (\det G)^2 \mathbf{1}$$
Simple Example

\[ S = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \]

\[
G = c U_3 U_1 + z (-s) s U_3 U_2 U_1 + z^2 ((-s) c s U_3 U_2 U_2 U_1 + \cdots
\]

\[
G^2 = c^2 (U_3 U_1)^2 + z^2 s^4 (U_3 U_2 U_1)^2
- z^2 s^2 c^2 (U_3 U_1 U_3 U_2 U_2 U_1 + U_3 U_2 U_2 U_1 U_3 U_1) + \cdots
\]

Write \( U_3 U_1 U_3 U_2 U_2 U_1 = (U_3 U_1)^2 (U_1^{-1} U_2 U_1)^2 \)

\[
\overline{G^2} = (-\frac{1}{2}) c^2 + z^2 (-\frac{1}{2}) s^4 - 2 (-\frac{1}{2}) s^2 c^2
\]

so

\[
\overline{G^\dagger G} = \cos^2 \theta + z^2 \sin^2 \theta
\]
Equivalent classical problem

\[
\begin{align*}
\text{Probability that trail from } e_{\text{in}} \text{ ends at } e_{\text{out}} &= c^2 + z^2 s^2 \\
&= 1 \text{ if } z = 1.
\end{align*}
\]
Proof for a general graph uses \textbf{supersymmetry}

- write \((1 - z \mathcal{U})^{-1}\) as a gaussian integral over variables \(b_R(e)\) and \(b_L(e)\) localized at the ends of each edge:

\[
G = \frac{\int \Pi_e[db_L(e)][db_R(e)]b_L(e_2)b_L^\dagger(e_1) e^{W_b}}{\int \Pi_e[db_L(e)][db_R(e)] e^{W_b}}
\]

where \(W_b = W_{\text{edge}} + W_{\text{node}}\) with

\[
W_{\text{edge}} = z \sum_e b_L^\dagger(e) U_e b_R(e)
\]

\[
W_{\text{node}} = \sum_n \sum_a \sum_{ij} b_{Ra}^* (e_i)(S_n)_{ij} b_{La}(e_j)
\]

and the integration is wrt the coherent state measure

\[
\int [db] = (1/\pi^2) \int e^{-b^\dagger b} \prod_a \text{dRe } b_a \text{ dIm } b_a
\]

- however in this form it is difficult to average over the \(U_e\)
• introduce corresponding anticommuting Grassmann variables $f_{R,L}(e), \bar{f}_{R,L}(e)$

with

$$\int [df] = \int d\bar{f} df e^{-\bar{f}f}$$

so that

$$\int [df] f = \int [df] \bar{f} = 0; \quad \int [df] 1 = \int [df] \bar{f} f = 1$$

Then

$$G(e_2, e_1) = \int \Pi_e [db_L(e)][db_R(e)][df_L(e)][df_R(e)] b_L(e_2)b_L^\dagger(e_1) e^{W_b+W_f}$$

• $W_b + W_f$ is invariant under global supersymmetry rotations.
Quenched average over the $U_e$

On each edge we have

$$\int dU \exp(z b_{L}^\dagger U b_{R} + z f_{L} U f_{R})$$

**Lemma:** this equals

$$1 +
\frac{1}{2} z^2 (b_{L1}^* f_{L2} - b_{L2}^* f_{L1})(b_{R1} f_{R2} - b_{R2} f_{R1})
+ z^2 (f_{L1} f_{L2})(f_{R2} f_{R1})$$

- interpretation: after averaging, all that can propagate along a given edge is:
  - $-1$, or
  - a fermion-boson pair $(1/\sqrt{2})(b_1 f_2 - f_1 b_2)$
  - a fermion-fermion pair $f_1 f_2$
• to compute $\mathcal{G}^2$, we can follow the propagation of (e.g.) a fermion-fermion pair

• it follows a *trail* through $\mathcal{G}$

• at a given node, we contract the incoming $\bar{f}_1 f_2$ pairs on edges $J$ onto the outgoing $f_1 f_2$ pairs on edges $I$ using Wick’s theorem

• this gives rise to the factors $\Omega(I; J)$ in Theorem 1.