Conformal restriction and the stress tensor of conformal field theory

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Motivation

- **SLE** and **CFT** are two different ways of describing the conjectured scaling limit of critical lattice $O(n)$ models:
  - **SLE** (and related ideas) describe the measure in the scaling limit of the random curves in these models;
  - **CFT** characterises the correlation functions: expectation values of products of local observables
- the underlying principles of SLE are more simply stated than those of CFT and appear to be easier to verify for particular lattice models
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Can we identify the local observables of CFT within SLE and show that they satisfy the rules of CFT?
QFT in a domain $\mathcal{D}$ is a collection of correlation functions:

- smooth maps from $(z_1, \ldots, z_N; z_i \in \mathcal{D}; z_i \neq z_j) \rightarrow \mathbb{C}$
- denoted by $\langle \phi_1(z_1) \ldots \phi_N(z_N) \rangle$
- satisfying closed operator product expansions (OPEs)

$$\phi_i(z_i) \cdot \phi_j(z_j) = \sum_k C_{ijk} (z_i - z_j) \phi_k((z_i + z_j)/2)$$

- physicists like to think of $\{\phi_j\}$ as local fields and $\langle \ldots \rangle$ as an $\mathbb{E}[\ldots]$ wrt some measure, but ???
Conformal field theory

- In CFT there are special local fields (primary) whose correlation functions transform simply under conformal transformations $z \to f(z)$:

$$
\phi(z, \bar{z}) \to f'(z)^{h\phi} f'(\bar{z})^{\bar{h}\phi} \phi(f(z), \bar{f}(\bar{z}))
$$

- Stress tensor $T(z)$ is holomorphic and generates infinitesimal conformal transformations $f(z) = z + \alpha(z)$ via insertion of

$$
\int_C \frac{dz}{2\pi i} \alpha(z) T(z) + \text{c.c.}
$$

into correlation functions

- Equivalent to conformal Ward identities

$$
\langle T(z) \prod_j \phi_j(z_j) \rangle = \sum_j \left[ \frac{h_{\phi_j}}{(z - z_j)^2} + \frac{1}{z - z_j} \frac{\partial}{\partial z_j} \right] \langle \prod_j \phi_j(z_j) \rangle
$$
or equivalently the OPE

\[ T(z) \cdot \phi_j(z_j) = \frac{h_{\phi_j}}{(z - z_j)^2} \phi_j(z_j) + \frac{1}{z - z_j} \partial z_j \phi(z_j) + \cdots \]

but this is modified if \( \phi_j = T \):

\[ T(z) \cdot T(z_j) = \frac{c/2}{(z - z_j)^4} + \frac{2}{(z - z_j)^2} T(z_j) + \frac{1}{z - z_j} \partial z_j T(z_j) + \cdots \]

\( c \) is the conformal anomaly (central charge)
Conformal restriction

measure on $\gamma$ restricted not to lie in $A$ is the same as the measure we get by conformally mapping $D \rightarrow D \setminus A$

more generally, if $D \setminus A$ is not simply connected, and $\Phi$ is a conformal map $D \rightarrow D'$, if $\mu$ is a restriction measure

$$\mu|_{\gamma \subset D' \setminus A'} = \Phi \circ \mu|_{\gamma \subset D \setminus A}$$
Conformal restriction

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- More generally, if $D \setminus A$ is not simply connected, and $\Phi$ is a conformal map $D \to D'$, if $\mu$ is a restriction measure
  \[
  \mu|_{\gamma \subset D' \setminus A'} = \Phi \circ \mu|_{\gamma \subset D \setminus A}
  \]

- [L–S–W] SLE$_\kappa$ satisfies this only if $\kappa = \frac{8}{3}$
- Conjectured to be the scaling limit of self-avoiding walks or the O(0) model: closed loops counted with weight 0
- how we identify the stress tensor $T$ (and other local operators) for these random curves?
- can we derive the conformal Ward identities?

In a fluid, $T_{\mu \nu} dS^\nu$ is the momentum flowing across $dS$. In a scale-invariant 2d system, $T_{\mu \mu} = 0$ and it has 2 independent components ($T$, $T$) which have 'spin' $\pm 2$: under $z \rightarrow z e^{i \theta}$, $T \rightarrow e^{-2i \theta} T$, $T \rightarrow e^{2i \theta} T$.

leads to the following guess: (inspired by Friedrich and Werner)
how we identify the stress tensor $T$ (and other local operators) for these random curves?

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what is the stress-energy tensor?

- in a fluid $T_{\mu\nu}dS^\nu$ is the momentum flowing across $dS^\nu$
- in a scale-invariant 2d system $T_\mu = 0$ and it has 2 independent components $(T, \overline{T})$ which have ‘spin’ $\pm 2$: under $z \rightarrow ze^{i\theta}$,
  $T \rightarrow e^{-2i\theta}T$, $\overline{T} \rightarrow e^{2i\theta}\overline{T}$

leads to the following guess: (inspired by Friedrich and Werner)
let $\gamma$ be an SLE$_{8/3}$ in $\mathbb{H}$ from $a_0$ to $\infty$

condition $\gamma$ on the event $E$ of passing to the L or R of marked points $\{\zeta_k\}$

slits of lengths $\{\epsilon_j\}$, at angles $\{\theta_j\}$, centred on points $\{z_j\}$

let

$$P(\{\epsilon_j\}, \{\theta_j\}, \{z_j\}; E(\{\zeta_k\}; a_0)) = \Pr(\gamma \text{ intersects every slit})$$

$$Q(\{z_j\}; \{\zeta_k\}; a_0) = \lim_{\epsilon_j \to 0} \prod_j \left(8/\pi \epsilon_j^2\right) \int \frac{d\theta_j}{2\pi} e^{-2i\theta_j} P(\ldots)$$
Theorem. [Doyon–Riva–JC]: the limit exists and satisfies

\[
Q(z_1, z_2, \ldots) = \left[ \sum_{j \neq 1} \frac{2}{(z_1 - z_j)^2} + \frac{1}{z_1 - z_2} \frac{\partial}{\partial z_j} \right.
\]

\[
+ \sum_k \frac{1}{z_1 - \zeta_k} \frac{\partial}{\partial \zeta_k} + \sum_k \frac{1}{z_1 - \bar{\zeta}_k} \frac{\partial}{\partial \bar{\zeta}_k}
\]

\[
+ \frac{5/8}{(z_1 - a_0)^2} + \frac{1}{z_1 - a_0} \frac{\partial}{\partial a_0} \left] Q(z_2, \ldots) \right.
\]

\[
\text{same as the conformal Ward identities satisfied by}
\]

\[
\langle T(z_1)T(z_2) \ldots \prod_k \Phi(\zeta_k)\phi(a_0)!\phi(\infty) \rangle
\]

with \( c = 0, h_\Phi = 0 \) and \( h_\phi = \frac{5}{8} \).
Idea of proof

- replace the small slit at $z_1$ by a small disc of radius $\epsilon_1$
- conformally map back to the slit using

$$g(z; \epsilon_1, \theta_1, z_1) = z - \frac{\epsilon_1^2}{16} \frac{e^{2i\theta_1}}{z - z_1} + \frac{\epsilon_1^2}{16} \frac{e^{2i\theta_1}}{a_0 - z_1} + \cdots$$

- this has an $O(\epsilon_1^2)$ effect on the parameters $\{\epsilon_j, \theta_j, z_j\}$ of the other slits (and also slightly deforms them, but to higher order in $\epsilon_1$)
now use the generalisation [Beffara, Lawler] to multiply connected domains of the [LSW] identity

$$P(\gamma \subset \mathbb{H} \setminus A) = g'(a_0)^{5/8}P(\gamma \subset \mathbb{H} \setminus g(A))$$

where

$$A = \text{disc at } z_1 \cup \text{various subsets of the other slits}$$

then do \( \int d\theta_1 e^{-2i\theta_1} \)

technical difficulty is in showing that the terms \( \cdots \) in \( g(z) \) as well as the higher order deformations of the slits can be neglected in the limit \( \epsilon_j \to 0 \)

amounts to an assertion that the probabilities \( P(\ldots) \) are sufficiently smooth in the moduli space of \( \mathbb{H} \setminus (\text{slits and the marked points } \{\zeta_k\}) \).
Werner has argued that there is a unique (up to a multiplicative constant) measure on self-avoiding loops in $\mathcal{D}$ satisfying conformal restriction conjectured to be the scaling limit of lattice self-avoiding loops. In order to apply our result we need to eliminate small loops, e.g., by restricting to loops which surround two distinct points $(\zeta_1, \zeta_2)$ in $\mathcal{D}$:

![Diagram of self-avoiding loops surrounding two distinct points.](image-url)
conjectured new result [Adam Gamsa, JC] from CFT:

let \( N(L; \zeta_1, \zeta_2) \) be the number of loops in \( \mathcal{D} \) of length \( L \) surrounding both points, then

\[
\lim_{\text{mesh} \to 0} \sum_L N(L; \zeta_1, \zeta_2) \mu^{-L} = -\frac{1}{12\pi} \ln(\eta(1 - \eta)) - \frac{1}{6\pi} \frac{\eta}{2} _3F_2(1, 1, 4/3; 2, 5/3, \eta) + \frac{\Gamma(2/3)^2}{6\pi \Gamma(4/3)} (-\eta(1 - \eta))^{1/3} _2F_1(2/3, 1, 4/3, \eta)
\]

where \( \eta \) is the modulus.

this corresponds to a correlation function \( \langle \Phi(\zeta_1)\Phi(\zeta_2) \rangle \) in CFT

if we now ask for the probability the loop intersects a slit \((z, \epsilon, \theta)\), take \( \int d\theta e^{-2i\theta} \) and scale with \((8/\pi\epsilon^2)\), we get the CFT Ward identity

\[
\langle T(z)\Phi(\zeta_1)\Phi(\zeta_2) \rangle = \sum_{k=1,2} \frac{1}{z - \zeta_k} \frac{\partial}{\partial \zeta_k} \langle \Phi(\zeta_1)\Phi(\zeta_2) \rangle
\]
Constructing the full CFT

- so far we have considered only events that either
  - $\gamma$ avoids a small region (e.g., a slit), or
  - $\gamma$ lies to one side or another of marked points $\{\zeta_k\}$
- by taking limits $\zeta_i \to \zeta_j$ and different Fourier components, and scaling with a suitable power of $\epsilon$, we can define many quantities which should correspond to correlators of local conformal fields
- they will all satisfy the correct conformal Ward identities with the $T$ we have identified
- next step is to show that they form a closed operator algebra among themselves
Beyond $c = 0$

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  - the $O(0)$ model
  - $\text{SLE}_\kappa$ with $\kappa < \frac{8}{3}$ + suitable density of Brownian bubbles
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- expect $T$ related to probability that any of the loops intersects slit
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- other values of the spin $s$ give holomorphic observables at other values of $\kappa = 8/(s + 1)$