Conformal restriction and the stress tensor of conformal field theory

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Motivation

- SLE and CFT are two different ways of describing the conjectured scaling limit of critical lattice $O(n)$ models:
  - SLE (and related ideas) describe the measure in the scaling limit of the random curves in these models;
  - CFT characterises the correlation functions: expectation values of products of local observables
- the underlying principles of SLE are more simply stated than those of CFT and appear to be easier to verify for particular lattice models

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Can we identify the local observables of CFT within SLE and show that they satisfy the rules of CFT?

$2d$ Quantum Field Theory

- QFT in a domain $\mathcal{D}$ is a collection of correlation functions:
  - smooth maps from $(z_1, \ldots, z_N; z_i \in \mathcal{D}; z_i \neq z_j) \to \mathbb{C}$
  - denoted by $\langle \phi_1(z_1) \ldots \phi_N(z_N) \rangle$
  - satisfying closed operator product expansions (OPEs)

$$\phi_i(z_i) \cdot \phi_j(z_j) = \sum_k C_{ijk}(z_i - z_j) \phi_k((z_i + z_j)/2)$$

- physicists like to think of $\{\phi_j\}$ as local fields and $\langle \ldots \rangle$ as an $\mathbb{E}[\ldots]$ wrt some measure, but ???

Conformal field theory

- in CFT there are special local fields (primary) whose correlation functions transform simply under conformal transformations $z \to f(z)$:

$$\phi(z,\bar{z}) \to f'(z)^{h_\phi}\,\overline{f'(z)}^{\bar{h}_\phi}\,\phi\big(f(z),\overline{f(z)}\big)$$

- stress tensor $T(z)$ is holomorphic and generates infinitesimal conformal transformations $f(z) = z + \alpha(z)$ via insertion of

$$\int_C \frac{dz}{2\pi i}\alpha(z)T(z) + \text{c.c.}$$

  into correlation functions

- equivalent to conformal Ward identities

$$\langle T(z)\prod_j \phi_j(z_j)\rangle = \sum_j \left[\frac{h_{\phi_j}}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right]\langle\prod_j \phi_j(z_j)\rangle$$

- or equivalently the OPE

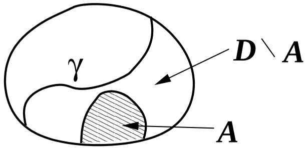
$$T(z) \cdot \phi_j(z_j) = \frac{h_{\phi_j}}{(z-z_j)^2}\phi_j(z_j) + \frac{1}{z-z_j}\partial_{z_j}\phi(z_j) + \cdots$$

- but this is modified if $\phi_j = T$:

$$T(z) \cdot T(z_j) = \frac{c/2}{(z-z_j)^4} + \frac{2}{(z-z_j)^2}T(z_j) + \frac{1}{z-z_j}\partial_{z_j}T(z_j) + \cdots$$

- c is the conformal anomaly (central charge)

Conformal restriction



- measure on γ restricted not to lie in A is the same as the measure we get by conformally mapping $\mathcal{D} \to \mathcal{D} \setminus A$
- more generally, if $\mathcal{D} \setminus A$ is not simply connected, and Φ is a conformal map $\mathcal{D} \to \mathcal{D}'$, if μ is a restriction measure

$$\mu|_{\gamma \subset \mathcal{D}' \setminus A'} = \Phi \circ \mu|_{\gamma \subset \mathcal{D} \setminus A}$$

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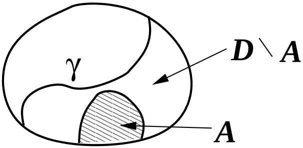

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- `[L-S-W]` SLE_κ satisfies this only if $\kappa = \frac{8}{3}$
- conjectured to be the scaling limit of self-avoiding walks or the $\mathrm{O}(0)$ model: closed loops counted with weight 0

- how we identify the stress tensor T (and other local operators) for these random curves?
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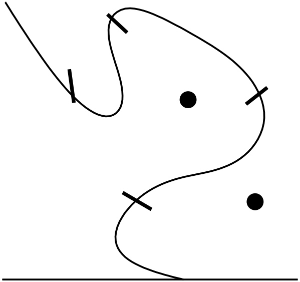
- how we identify the stress tensor T (and other local operators) for these random curves?
- can we derive the conformal Ward identities?
- what is the stress-energy tensor?
  - in a fluid $T_{\mu\nu}dS^\nu$ is the momentum flowing across dS^ν
  - in a scale-invariant $2d$ system $T^\mu_\mu = 0$ and it has 2 independent components $(T, \overline{T})$ which have 'spin' ± 2: under $z \to ze^{i\theta}$, $T \to e^{-2i\theta}T, \overline{T} \to e^{2i\theta}\overline{T}$
- leads to the following guess: (inspired by `Friedrich and Werner`)



- let γ be an $\mathrm{SLE}_{8/3}$ in $\mathbb{H}$ from a_0 to ∞
- condition γ on the event E of passing to the L or R of marked points $\{\zeta_k\}$
- slits of lengths $\{\epsilon_j\}$, at angles $\{\theta_j\}$, centred on points $\{z_j\}$
- let

$$P\big(\{\epsilon_j\},\{\theta_j\},\{z_j\};E(\{\zeta_k\};a_0)\big) = \Pr\big(\gamma \text{ intersects every slit}\big)$$

$$Q\big(\{z_j\};\{\zeta_k\};a_0\big) = \lim_{\epsilon_j\to 0}\prod_j (8/\pi\epsilon_j^2)\int \frac{d\theta_j}{2\pi} e^{-2i\theta_j} P(\ldots)$$

Theorem. [Doyon-Riva-JC]: the limit exists and satisfies

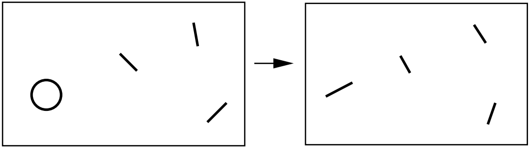
$$
\begin{aligned}
Q(z_1, z_2, \ldots) \quad = \quad & \Bigg[\sum_{j \neq 1} \frac{2}{(z_1 - z_j)^2} + \frac{1}{z_1 - z_2} \frac{\partial}{\partial z_j} \\
& + \sum_k \frac{1}{z_1 - \zeta_k} \frac{\partial}{\partial \zeta_k} + \sum_k \frac{1}{z_1 - \bar{\zeta}_k} \frac{\partial}{\partial \bar{\zeta}_k} \\
& + \frac{5/8}{(z_1 - a_0)^2} + \frac{1}{z_1 - a_0} \frac{\partial}{\partial a_0} \Bigg] Q(z_2, \ldots)
\end{aligned}
$$

- same as the conformal Ward identities satisfied by

$$
\langle T(z_1) T(z_2) \ldots \prod_k \Phi(\zeta_k) \phi(a_0) \phi(\infty) \rangle
$$

with $c = 0$, $h_\Phi = 0$ and $h_\phi = \frac{5}{8}$.

Idea of proof



- replace the small slit at z_1 by a small disc of radius ϵ_1
- conformally map back to the slit using

$$g(z;\epsilon_1,\theta_1,z_1) = z - \frac{\epsilon_1^2}{16}\frac{e^{2i\theta_1}}{z-z_1} + \frac{\epsilon_1^2}{16}\frac{e^{2i\theta_1}}{a_0-z_1} + \cdots$$

- this has an $O(\epsilon_1^2)$ effect on the parameters $\{\epsilon_j,\theta_j,z_j\}$ of the other slits (and also slightly deforms them, but to higher order in ϵ_1)

- now use the generalisation [Beffara,Lawler] to multiply connected domains of the [LSW] identity

$$P\big(\gamma \subset \mathbb{H} \setminus A\big) = g'(a_0)^{5/8} P\big(\gamma \subset \mathbb{H} \setminus g(A)\big)$$

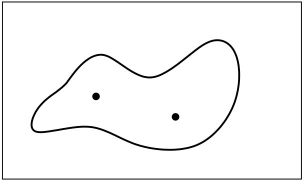
where

$$A = \text{disc at } z_1 \cup \text{various subsets of the other slits}$$

- then do $\int d\theta_1 e^{-2i\theta_1}$
- technical difficulty is in showing that the terms $\cdots$ in $g(z)$ as well as the higher order deformations of the slits can be neglected in the limit $\epsilon_j \to 0$
- amounts to an assertion that the probabilities $P(\ldots)$ are sufficiently smooth in the moduli space of $\mathbb{H} \setminus (\text{slits and the marked points } \{\zeta_k\})$.

Other restriction measures

- `Werner` has argued that there is a unique (up to a multiplicative constant) measure on self-avoiding loops in $\mathcal{D}$ satisfying conformal restriction
- conjectured to be the scaling limit of lattice self-avoiding loops
- in order to apply our result we need to eliminate small loops, eg by restricting to loops which surround two distinct points (ζ_1, ζ_2) in $\mathcal{D}$:



- conjectured new result `[Adam Gamsa, JC]` from CFT:
- let $N(L;\zeta_1,\zeta_2)$ be the number of loops in $\mathcal{D}$ of length L surrounding both points, then

$$
\begin{aligned}
&\lim_{\text{mesh}\to 0}\sum_L N(L;\zeta_1,\zeta_2)\mu^{-L}\\
=\quad & -\frac{1}{12\pi}\ln(\eta(1-\eta)) - \frac{1}{6\pi}\frac{\eta}{2}\,{}_3F_2(1,1,4/3;2,5/3,\eta)\\
&+\frac{\Gamma(2/3)^2}{6\pi\Gamma(4/3)}(-\eta(1-\eta))^{\frac{1}{3}}\,{}_2F_1(2/3,1,4/3,\eta)
\end{aligned}
$$

where η is the modulus.

- this corresponds to a correlation function $\langle\Phi(\zeta_1)\Phi(\zeta_2)\rangle$ in CFT
- if we now ask for the probability the loop intersects a slit (z,ϵ,θ), take $\int d\theta e^{-2i\theta}$ and scale with $(8/\pi\epsilon^2)$, we get the CFT Ward identity

$$
\langle T(z)\Phi(\zeta_1)\Phi(\zeta_2)\rangle = \sum_{k=1,2}\frac{1}{z-\zeta_k}\frac{\partial}{\partial\zeta_k}\langle\Phi(\zeta_1)\Phi(\zeta_2)\rangle
$$

Constructing the full CFT

- so far we have considered only events that either
  - γ avoids a small region (eg a slit), or
  - γ lies to one side or another of marked points $\{\zeta_k\}$
- by taking limits $\zeta_i \to \zeta_j$ and different Fourier components, and scaling with a suitable power of ϵ, we can define many quantities which should correspond to correlators of local conformal fields
- they will all satisfy the correct conformal Ward identities with the T we have identified
- next step is to show that they form a closed operator algebra among themselves

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- expect T related to probability that any of the loops intersects slit
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- other values of the spin s give holomorphic observables at other values of $\kappa = 8/(s+1)$