CONFORMAL INVARIANCE in PERCOLATION SELF-AVOIDING WALKS and related problems

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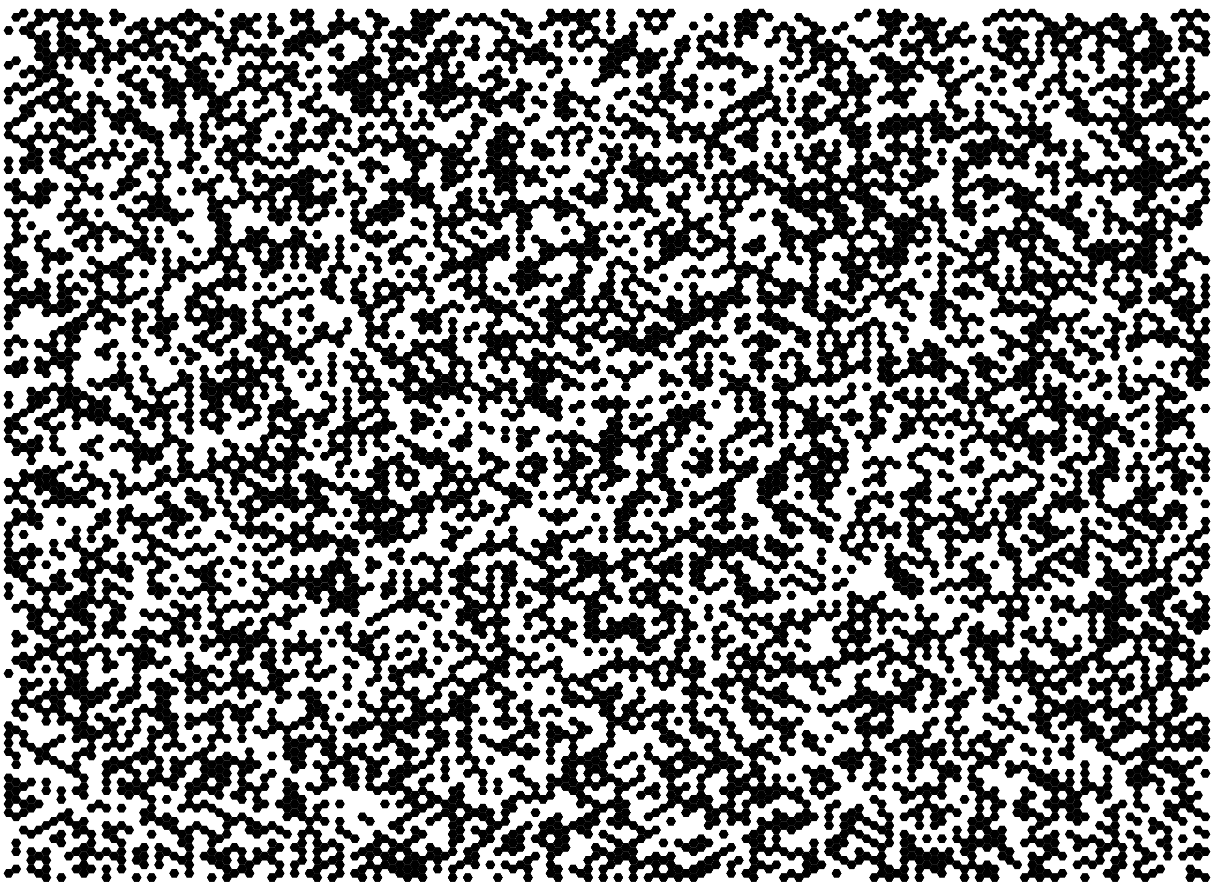
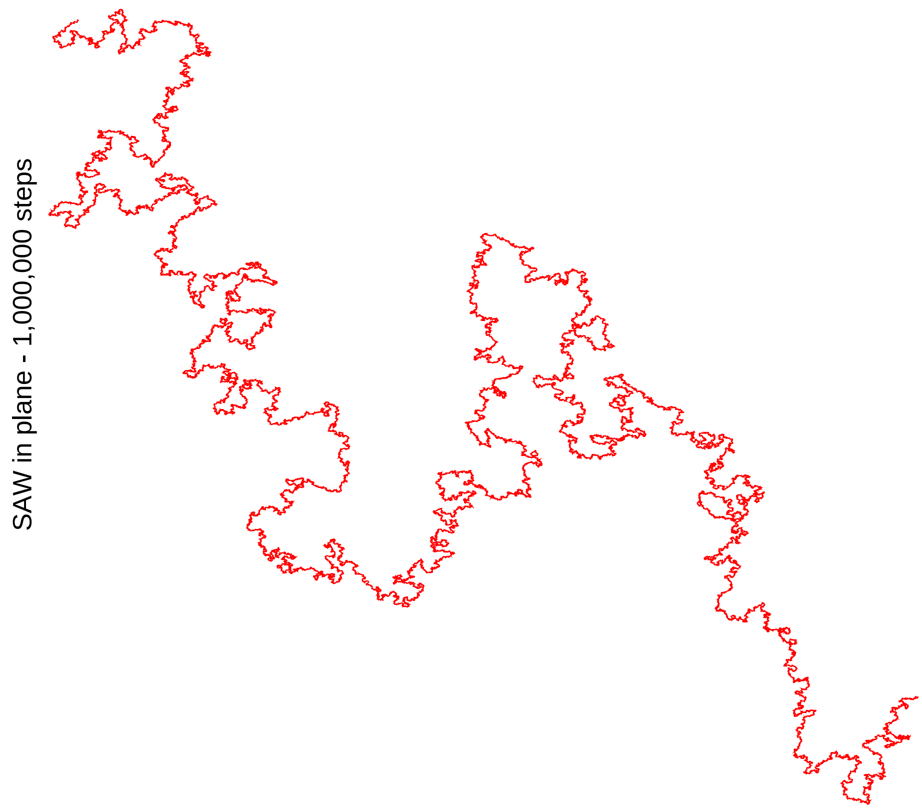
‘Geometric’ critical phenomena: random spatial processes where either:

- we ask questions about geometrical properties, eg clustering in **percolation**:



or

• the probability distribution is geometrical in nature (eg **self-avoiding walks** of a fixed length, all weighted equally)



• percolation relevant to disordered media

• SAWs relevant but to polymer physics

But emphasis in this talk is on understanding the nature of their fractal geometry.

OUTLINE

- expectations for ‘geometric’ critical behaviour from conventional critical behaviour;
- some (non-rigorous) results from (conformal) field theory (in 2D);
- cluster boundaries and Coulomb gas methods;
- direct construction of continuum limit of cluster boundaries (SLE);
- rigorous and new results for percolation;
- some provocative conclusions

‘Geometric’ *vs.* ‘conventional’ critical behaviour

- [Fortuin-Kastelyn] Q-state Potts model $\Leftrightarrow$ Random Cluster Model:

$$Z = \langle Q^{|\text{clusters}|} \rangle_{\text{percolation}}$$

$Q \to 1 \Leftrightarrow$ percolation.

- [de Gennes] $\mathrm{O}(n)$ model $\Leftrightarrow$ Self-Avoiding Loops:

$$Z = \langle n^{|\text{loops}|} \rangle_{\text{loop gas}}$$

$n = 1 \Leftrightarrow$ Ising model;

$n \to 0$ single self-avoiding loop.

Dictionary Examples

Cluster size $\Leftrightarrow$ susceptibility $\propto (p - p_c)^{-\gamma(Q)}$

Radius $\Leftrightarrow$ correlation length $\sim \text{mass}^{\nu(n)}$

Critical Behaviour & Euclidean Field Theory

• Near-critical lattice model, correlation length $\xi \gg$ lattice spacing a: *continuum limit*

$$\langle\phi(r_1)\ldots\phi(r_N)\rangle_{\mathrm{QFT}}$$
$$= \lim_{a\to 0,\xi\ \mathrm{fixed}} a^{-nx_\phi}\langle S(r_1)\ldots S(r_N)\rangle_{\mathrm{lattice}}$$

• emphasis on correlation functions of *local* (or quasi-local) operators and their algebra encoded in the OPE.

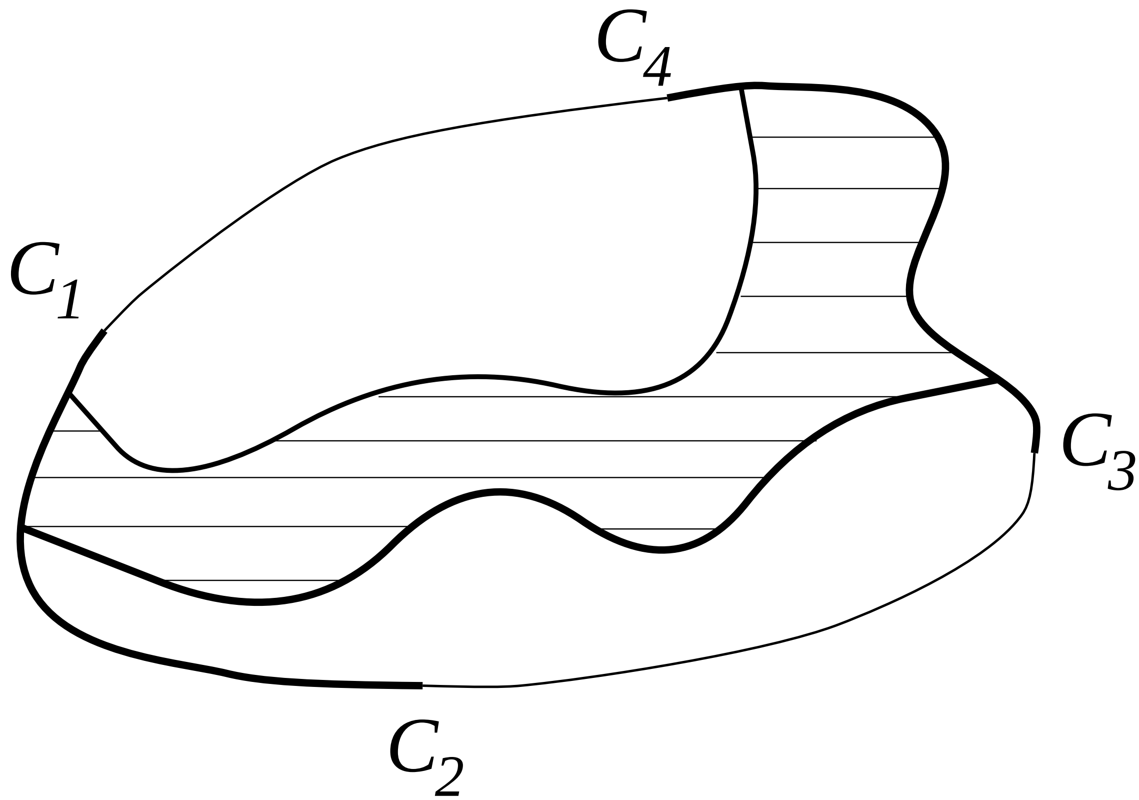
• *proved* in very few examples, but if *assumed* has many powerful consequences: RG, universality, and **scaling**:
off-critical behaviour $\leftrightarrow$
decay of correlation functions *at* critical point
= conformal field theory.

Conformal Field Theory

- in local *classical* field theories, scale invariance $\Rightarrow T^\mu_\mu = 0 \Rightarrow$ conformal invariance.
- CFT assumes this holds (up to conformal anomaly c) in the theory including fluctuation effects.
- Operators $\Leftrightarrow$ States correspondence.
- For $d = 2$, these transform according to reps of infinite-dimensional Virasoro algebra.
- Classification problem: given some critical lattice model, to which CFT does it correspond?
- [Friedan,Qiu,Shenker] Reflection positivity (eg. $Q = 2,3,4$ or $n = 1,2$) $\Rightarrow$ *unitary* reps $\Rightarrow$ discrete series; null states $\Rightarrow$ linear differential equations for correlation functions [Belavin,Polyakov,Zamolodchikov].

- But percolation, SAWs, and related models are not unitary: in fact they have $Z = 1$ ($c = 0$) even though correlations are nontrivial!
- non-unitary $c = 0$ CFTs are very poorly understood but are important not just for percolation and SAWs, but for all critical problems with quenched disorder (e.g. quantum Hall transition)

The Crossing Formula in Percolation



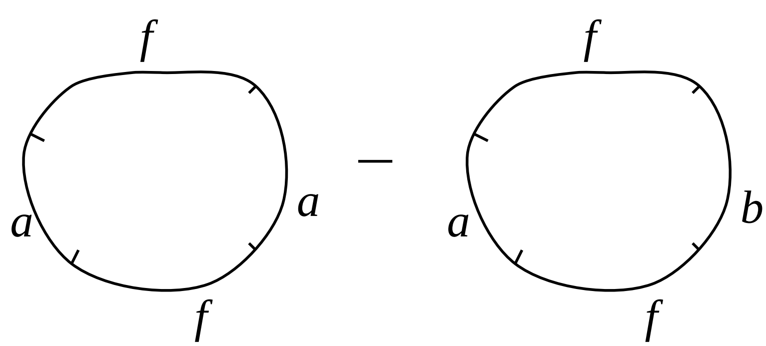
• Given a simply connected region D of the plane, with suitably smooth boundary ∂D, with 4 marked points C_j, what is the probability of a spanning cluster connecting C_1C_2 with C_3C_4 (in the limit lattice spacing $\to 0$)?

• CFT Conjecture [Cardy 1992] : conformally map interior of D into unit disc, marked points $C_j \to z_j$. Crossing probability depends only anharmonic ratio $\eta = z_{12}z_{34}/z_{13}z_{24}$ and is

$$\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{\frac{1}{3}}{}_2F_1(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};\eta)$$

Argument depends on:

- assuming scaling continuum limit exists and is given by a CFT with $c = 0$;
- realising that crossing probability is related to a difference of partition functions of the $Q \to 1$ Potts model, with different boundary conditions;



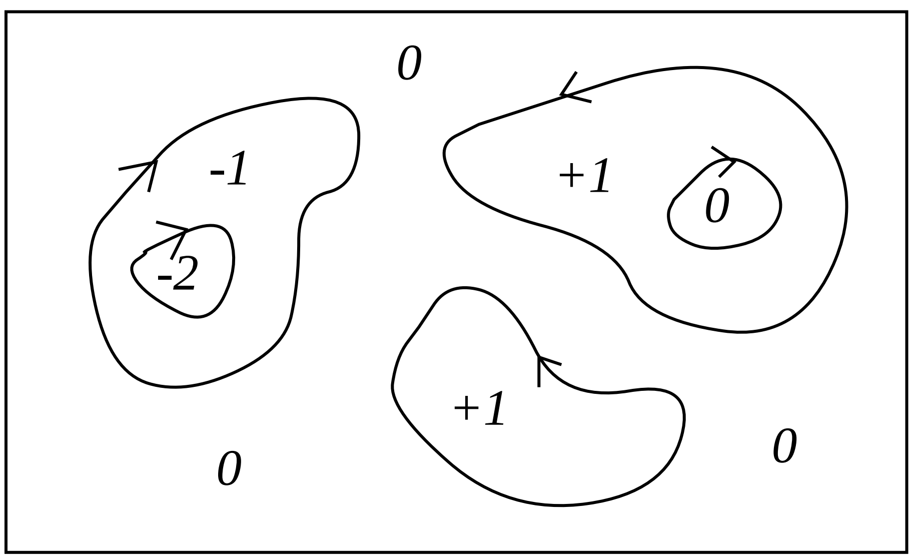
- realising that states of the CFT induced by changes in the boundary condition ("boundary condition changing operators") also should correspond to Virasoro reps;
- guessing the right rep $\Rightarrow$ differential equation.

• formula numerically verified to high precision, but hard to see how to make arguments rigorous, or to go beyond them.

Cluster Boundary Approach

1) **'Coulomb gas' method** [`den Nijs, Nienhuis, Duplantier, Saleur, Kondev, ...`]

- cluster boundaries in random cluster model (or loops in O(n) model) form a gas of closed loops.
- randomly orientate $\Rightarrow$ height model $h(r) \in$ integers:



- factors of Q (resp. n) can be associated with local (but complex!) Boltzmann weights;

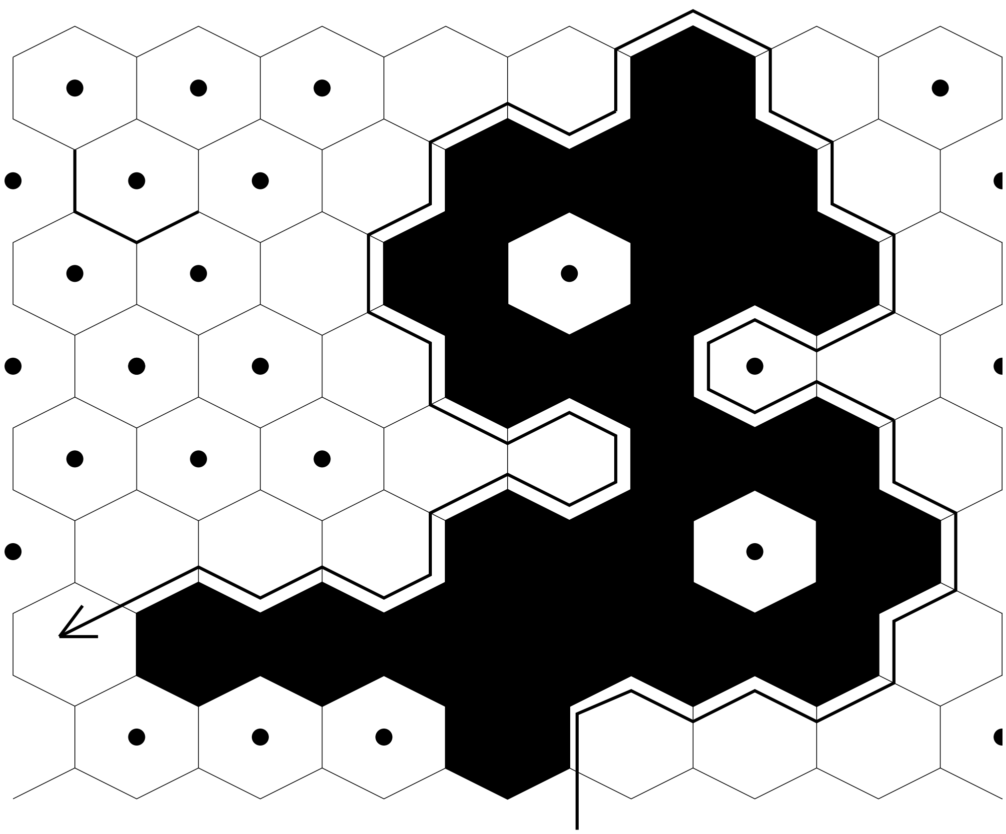
- *assume* in continuum limit that $h \in \mathbf{R}$, and measure $\to \exp(-g\int(\partial h)^2 d^2r)$;
- a $c = 1$ CFT, with, however, charge at ∞, screening charges, etc., which make it non-trivial;
- critical exponents calculable, eg $\nu_{\rm SAW} = \frac{3}{4}$ [`Nienhuis`] and $\nu_{\rm perc} = \frac{4}{3}$ [`den Nijs`];
- *but* correlation functions ambiguous, hard to make rigorous.
- nevertheless still new results: eg distribution of *internal areas* of loops: density $n(A)$ of large loops with area $> A$

$$n(A) \sim C/A \qquad C \text{ universal}$$

where [`Cardy,Ziff`]

$$\begin{aligned} C_{\rm perc} = 1/8\sqrt{3}\pi \;&=\; 0.0229720 \qquad \text{predicted} \\ &=\; 0.022972(1) \quad \text{measured} \end{aligned}$$

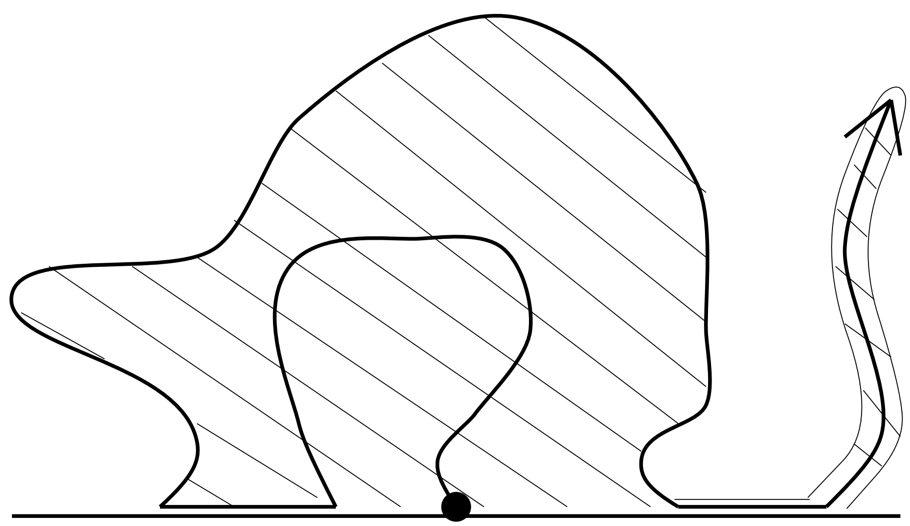
2) **Dynamical description of cluster boundaries**



- random walker lays down the configuration as it moves: black sites to the L, white to the R;
- the path automatically reflects from itself;
- **how to characterise the continuum limit of these paths?**

STOCHASTIC LOEWNER EVOLUTION (SLE) [Schramm; Lawler, Werner]

For definiteness, consider the half-plane with black sites along $x < 0$ and white sites along $x > 0$. Path starts at origin:



• rather than writing an equation for the path, consider the (unique) conformal mapping $z \to g(z;t)$ which sends {region of the half-plane which has not been excluded by the path} $\to$ {upper half-plane}.

- dynamics on conformal mappings:

$$\frac{\partial g(z;t)}{\partial t} = \frac{2}{g(z;t) - a(t)}$$

- if $a(t)$ is real continuous function, the excluded region of the half-plane grows with increasing t;

- $a(t)$ must be Brownian motion, ie $\dot{a} = \zeta(t)$ with $\overline{\zeta(t)\zeta(t')} = \kappa\delta(t-t')$.

- [Rohde & Schramm]

$$\begin{array}{rrl} \text{If} & 0 \leq \kappa \leq 4 & \text{path is simple} \\ & 4 < \kappa < 8 & \text{it touches itself} \\ & 8 < \kappa & \text{it fills space} \end{array}$$

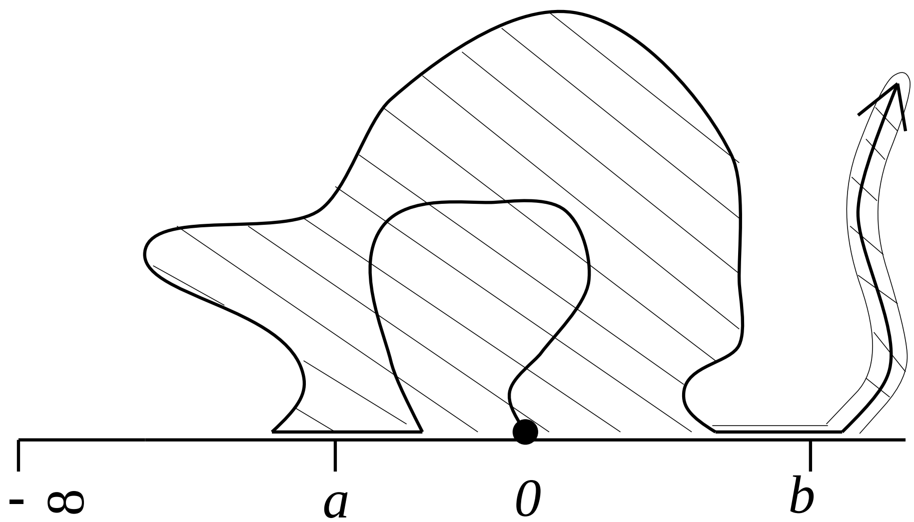
- fractal dimension $= 1 + \kappa/8$ (for $\kappa < 8$).

- *only* for $\kappa = 6$ does the SLE path not 'feel' where the boundary of the domain is as long as it does not hit it $\Rightarrow$ Conjecture [Schramm]

SLE_6 is the conformally invariant continuum limit of percolation cluster boundaries

- one can *compute* with SLE all previously conjectured critical exponents at p_c (and with the help of rigorous scaling relations [Kesten], exponents away from p_c) (including multi-fractal irrational but algebraic exponents (related to 2d quantum gravity [Duplantier talk])) and some new ones, eg:
- *backbone exponent* [Lawler, Schramm, Werner] (fractal dimension of the part of the infinite cluster which would carry electric current) – given by the lowest eigenvalue of a 2d Dirichlet problem (and probably not a rational or an algebraic number!)

• SLE and the crossing formula:



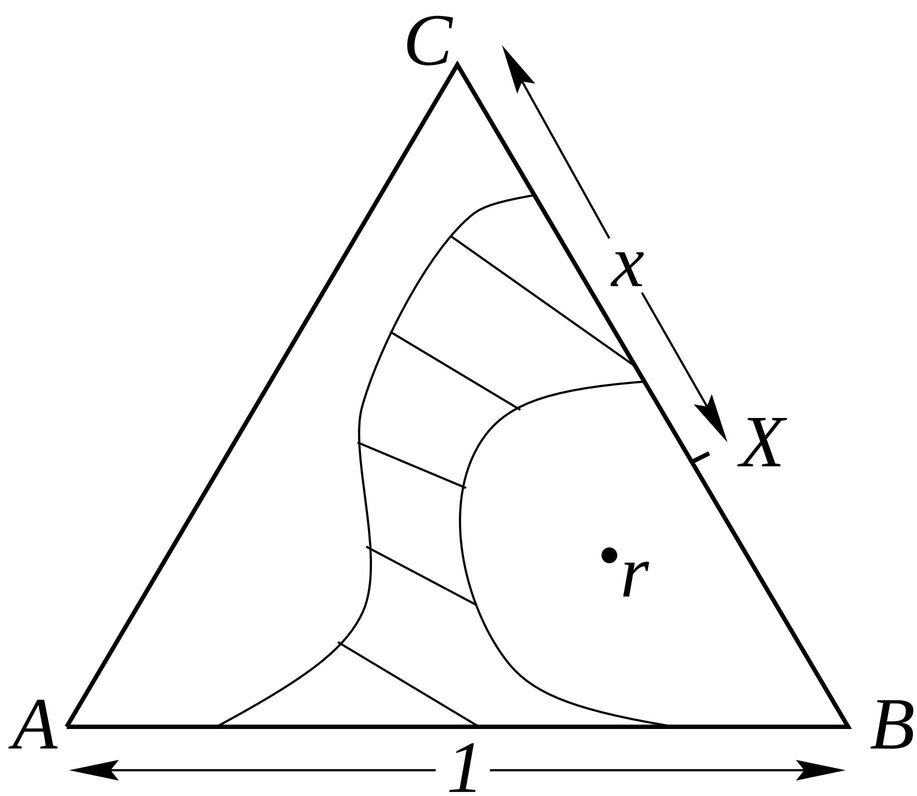
$$Pr(\text{white crossing } (-\infty, a) \leftrightarrow (0, b)$$
$$= Pr(a \text{ gets excluded before } b)$$

• this gives the same ${}_2F_1$ formula as conjectured from CFT.

The missing link: Smirnov's proof of the crossing formula

Smirnov proved that the crossing formula holds for the continuum limit of site percolation on a triangular lattice $\Rightarrow$ SLE$_6$ *is* the continuum limit of percolation cluster boundaries $\Rightarrow$ all the results derived from SLE$_6$ are *rigorous*.

- [`Carleson`] crossing formula is simple in an equilateral triangle



- it is the boundary value of an analytic function $P(z)$: what is the interpretation for z not on boundary?

  $P(z) = Pr(z$ separated from C_1C_4 by at least one cluster spanning from C_1C_2 to $C_2C_4)$

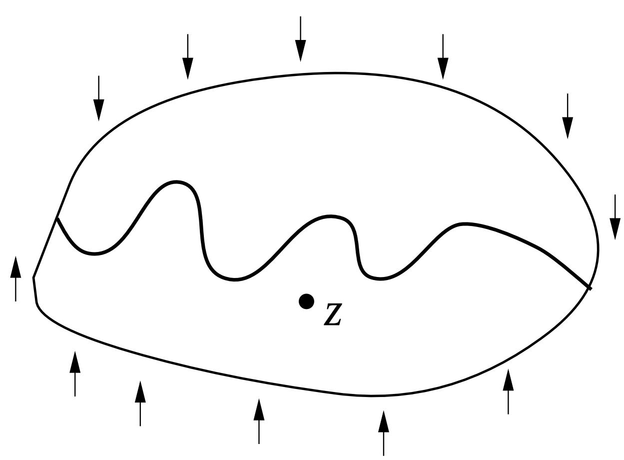
- [Smirnov] on the triangular lattice, $P(z)$ satisfies linear relations $\Rightarrow$ itscontinuum limit exists, and is the real part of a harmonic function.

  Boundary conditions then determine $P(z) \propto$ distance from $C_1C_4 \Rightarrow$ crossing formula.

Other values of κ

- Self-Avoiding Walks: uniform measure on set of simple paths must remain uniform when restricted to a subset. *If* the continuum limit is SLE_κ, then $\kappa = \frac{8}{3} \Rightarrow$ all the conjectured results for SAWs (and more) [Lawler, Schramm, Werner].

- Ising model: conjectured to be $\mathrm{SLE}_3 \Rightarrow$ all the standard results and some new ones: eg Pr(domain wall passes above z).



- **but** these results need analogue of Smirnov's proof to be made rigorous.

FINAL REMARKS

Progress in rigorous results for critical exponents:

- **1944** *ff*: 2d Ising model [Onsager, Yang, ...]

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- 1963 *ff*: many exact, but non-rigorous lattice results [Baxter,...]

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- 2001: rigorous results for 2d percolation [Kesten,Smirnov,Lawler,Schramm,Werner,...]

Why are rigorous results important?

Progress in QFT and critical behaviour:

- **1969** *ff*: Renormalisation group [`Wilson, `…]: – a non-rigorous framework in which to understand many important features of critical phenomena, and do approximate computations
- 1984 *ff*: Conformal field theory: – a plethora of exact but non-rigorous results in 2D
- **but**: field theory still tied to particle physics ideas: correlation functions of (quasi-)local fields, so is ill-suited to study other objects (eg crossing formula)
- has QFT outlived its usefulness to generate new results?
- OR can it reinvent itself in the 21st century:

QFT as fractal geometry?