

Logarithmic CFTs as limits of ordinary CFTs and some physical applications

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- ▶ mostly [arXiv:cond-mat/9911024](https://arxiv.org/abs/cond-mat/9911024), [0111031](https://arxiv.org/abs/cond-mat/0111031) and some new stuff

Introduction

- ▶ this talk will be physical rather than mathematical: how can logCFTs arise in physics?
- ▶ viewpoint will be to regard logCFTs as limits of ordinary (albeit irrational) CFTs
- ▶ most of the discussion not restricted to two dimensions, although some detailed examples will be
- ▶ two main examples:
 - ▶ random systems using the $n \rightarrow 0$ ‘replica trick’
 - ▶ self-avoiding walks as the $n \rightarrow 0$ limit of the $O(n)$ model
- ▶ both these have partition function $Z = 1$ ($c = 0$), and one of the interesting questions is how t , the logarithmic partner of the stress tensor T , emerges in this picture
- ▶ I will not address logCFTs with $c \neq 0$ in this talk

LogCFTs in general

- ▶ a logCFT is a scale-invariant QFT ($\Theta = T_{\mu}^{\mu} = 0$) where the generator \hat{S} of scale transformations cannot be completely diagonalized, but only brought to Jordan form
- ▶ e.g. for a 2×2 cell

$$\hat{S} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 1 & \Delta \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

- ▶ this has the consequence that (for scalar fields)

$$\langle D(r_1)D(r_2) \rangle \sim \frac{-2\alpha \log |r_1 - r_2| + O(1)}{|r_1 - r_2|^{2\Delta}}$$

$$\langle C(r_1)D(r_2) \rangle \sim \frac{\alpha}{|r_1 - r_2|^{2\Delta}}$$

$$\langle C(r_1)C(r_2) \rangle = 0$$

where α depends on how D is normalised.

Quenched random systems and replicas

- ▶ classical stat mech system, energy $E[\{\phi\}, \{h\}]$: $\{\phi(r)\}$ are fluctuating degrees of freedom; $\{h(r)\}$ are *quenched* random fields drawn from some distribution, e.g.

$$E[\{\phi\}, \{h\}] = E_{\text{pure}}[\{\phi\}] + \lambda \int h(r)\Phi(r)d^d r$$

where Φ is some local field and $\overline{h(r)} = 0$,
 $\overline{h(r)h(r')} = \delta^{(d)}(r - r')$.

- ▶ we want to compute quenched averages of correlators of local fields, e.g.

$$\overline{\langle \Phi(r_1)\Phi(r_2) \rangle} = \overline{\left(\frac{\text{Tr}_\phi \Phi(r_1)\Phi(r_2)e^{-E[\{\phi\}, \{h\}]}}{\text{Tr}_\phi e^{-E[\{\phi\}, \{h\}]}} \right)}$$

- ▶ this is difficult because of the $\{h\}$ -dependence in the denominator $Z[\{h\}]$

Two ways around this

- ▶ (a) find some other degrees of freedom $\{\psi\}$ such that $\text{Tr}_\psi e^{-E[\{\psi\},\{h\}]} = Z[\{h\}]^{-1}$, then

$$\overline{\langle \Phi(r_1)\Phi(r_2) \rangle} = \overline{\text{Tr}_{\phi,\psi} \Phi(r_1)\Phi(r_2) e^{-E[\{\phi\},\{h\}] - E[\{\psi\},\{h\}]}}$$

The quenched average is now easy. In some cases this leads to a *supersymmetry* between $\{\phi\}$ and $\{\psi\}$.

- ▶ (b) consider n copies of the fields $\{\phi_a\}$, $a = 1, \dots, n$, and note that

$$\overline{\langle \Phi(r_1)\Phi(r_2) \rangle} = \overline{\left(\frac{\text{Tr}_{\phi_a} \Phi_1(r_1)\Phi_1(r_2) e^{-\sum_{a=1}^n E[\{\phi_a\},\{h\}]} }{Z[\{h\}]^n} \right)}$$

for all integer $n \geq 1$.

- ▶ if this can be continued to $n = 0$, we can set the denominator = 1, the quenched average is again easy, but now the replicas interact.

Replica group theory

- ▶ the symmetry group is S_n
- ▶ for $\lambda = 0$, this acts trivially on the non-interacting replicas
- ▶ for $\lambda \neq 0$, assume that under the RG the theory flows to a CFT, which, for $n \neq 0$, is generically non-logarithmic
- ▶ eigenstates of \hat{D} transform according to *irreducible* representations of S_n
- ▶ in particular, the multiplet (Φ_1, \dots, Φ_n) decomposes into

$$\Phi \equiv \sum_{a=1}^n \Phi_a$$

$$\tilde{\Phi}_a \equiv \Phi_a - (1/n) \sum_{a=1}^n \Phi_a$$

Operator product expansions

- ▶ in the non-interacting theory we have the OPE (suppressing indices)

$$\Phi_a(r_1) \cdot \Phi_b(r_2) = \delta_{ab} |r_{12}|^{-2\Delta_{\text{pure}}} \left(1 + \dots + \frac{\Delta_{\text{pure}}}{c_{\text{pure}}} r_{12}^d T_a(r_1) + \dots \right)$$

- ▶ the coefficient of T is fixed by global conformal invariance and is valid in any number of dimensions, if c is defined as the coefficient of the 2-point function $\langle T(r_1)T(r_2) \rangle \sim c/r_{12}^{2d}$ (suppressing indices).

These are equivalent to

$$\begin{aligned}\tilde{\Phi}_a \cdot \tilde{\Phi}_a &= \left(1 - \frac{1}{n}\right) |r_{12}|^{-2\Delta_{\text{pure}}} \left(1 + \frac{\Delta_{\text{pure}}}{nc_{\text{pure}}} r_{12}^d T + \frac{\Delta_{\text{pure}}}{c} r_{12}^d \tilde{T}_a + \dots\right) \\ \Phi \cdot \Phi &= n |r_{12}|^{-2\Delta_{\text{pure}}} \left(1 + \frac{\Delta_{\text{pure}}}{nc_{\text{pure}}} r_{12}^d T + \dots\right)\end{aligned}$$

where T and \tilde{T}_a are the corresponding irreducible linear combinations of the T_a .

In the interacting theory these deform to

$$\begin{aligned}\tilde{\Phi}_a \cdot \tilde{\Phi}_a &= \left(1 - \frac{1}{n}\right) |r_{12}|^{-2\tilde{\Delta}(n)} \left(1 + \frac{\tilde{\Delta}(n)}{c(n)} r_{12}^d T + B(n) r_{12}^{d+\delta(n)} \tilde{T}_a + \dots\right) \\ \Phi \cdot \Phi &= n |r_{12}|^{-2\Delta(n)} \left(1 + \frac{\Delta(n)}{c(n)} r_{12}^d T + \dots\right)\end{aligned}$$

where $(\Delta(n), \tilde{\Delta}(n), d, d + \delta(n))$ are the dimensions of $(\Phi, \tilde{\Phi}_a, T, \tilde{T}_a)$ respectively and $c(n)$ is the central charge of the interacting theory.

If we look at the 2-point functions we have

$$\langle \Phi_1(r_1)\Phi_1(r_2) \rangle = \langle \tilde{\Phi}_1\tilde{\Phi}_1 \rangle + \frac{1}{n^2}\langle \Phi\Phi \rangle = \left(1 - \frac{1}{n}\right) \frac{1}{|r_{12}|^{2\tilde{\Delta}(n)}} + \frac{1}{n} \frac{1}{|r_{12}|^{2\Delta(n)}}$$

$$\langle \Phi_1(r_1)\Phi(r_2) \rangle = \frac{1}{n}\langle \Phi\Phi \rangle = \frac{1}{|r_{12}|^{2\Delta(n)}}$$

$$\langle \Phi(r_1)\Phi(r_2) \rangle = \frac{n}{|r_{12}|^{2\Delta(n)}}$$

But

$$\lim_{n \rightarrow 0} \langle \Phi_1(r_1)\Phi_1(r_2) \rangle = \overline{\langle \Phi(r_1) \rangle \langle \Phi(r_2) \rangle}$$

$$\lim_{n \rightarrow 0} \langle \Phi_1(r_1)\Phi(r_2) \rangle = \overline{\langle \Phi(r_1)\Phi(r_2) \rangle - \langle \Phi(r_1) \rangle \langle \Phi(r_2) \rangle}$$

and these had better be finite!

- ▶ as $n \rightarrow 0$, $\tilde{\Delta}(n) - \Delta(n) \rightarrow 0$, and

$$\langle \Phi_1(r_1)\Phi_1(r_2) \rangle \sim \frac{-2\alpha \log |r_{12}|}{|r_{12}|^{2\Delta(0)}} \quad \text{where} \quad \alpha = \tilde{\Delta}'(0) - \Delta'(0)$$

- ▶ (Φ_1, Φ) are an example of a *logarithmic pair*.

The ‘ $c \rightarrow 0$ catastrophe’

$$\tilde{\Phi}_a \cdot \tilde{\Phi}_a = \left(1 - \frac{1}{n}\right) |r_{12}|^{-2\tilde{\Delta}(n)} \left(1 + \frac{\tilde{\Delta}(n)}{c(n)} r_{12}^d T + B(n) r_{12}^{d+\delta(n)} \tilde{T}_a + \dots\right)$$

$$\Phi \cdot \Phi = n |r_{12}|^{-2\Delta(n)} \left(1 + \frac{\Delta(n)}{c(n)} r_{12}^d T + \dots\right)$$

- ▶ since $\lim_{n \rightarrow 0} c(n) = 0$, there is an apparent problem with the coefficient of T . This can be resolved in several different ways:
 - a) the normalization of the physical fields vanishes as $c \rightarrow 0$: this is what happens for $\Phi \cdot \Phi$ above (and for many examples in percolation): no logs
 - b) the scaling dimension $\Delta \rightarrow 0$: this happens for the Kac (1,2) operator in 2d percolation whose 4-pt function gives the crossing formula: no logs
 - c) T collides with another operator as $c \rightarrow 0$, and the leading singularities cancel: this is what happens for $\tilde{\Phi}_a \cdot \tilde{\Phi}_a$ above, as long as $\delta(n) \rightarrow 0$ and $B(n) \sim 2\Delta(n)/c(n)$. In this case we are left with a term $\propto \delta'(0) r_{12}^d \log |r_{12}|$.

In this case T_1 and T form a logarithmic pair: in 2d

$$\langle T_1(z_1)T_1(z_2) \rangle = \overline{\langle T(z_1) \rangle \langle T(z_2) \rangle} = \frac{2c'(0)\delta'(0) \log(z_{12}\bar{z}_{12})}{2z_{12}^4}$$

$$\langle T_1(z_1)T(z_2) \rangle = \overline{\langle T(z_1)T(z_2) \rangle - \langle T(z_1) \rangle \langle T(z_2) \rangle} = \frac{c'(0)}{2z_{12}^4}$$

$$\langle T(z_1)T(z_2) \rangle = 0$$

In 2d logCFT this is usually written, with $t \propto T_1$,

$$\langle t(z_1)t(z_2) \rangle = -\frac{b \log(z_{12}\bar{z}_{12})}{z_{12}^4}$$

$$\langle t(z_1)T(z_2) \rangle = \frac{b}{2z_{12}^4}$$

so that $b = -c'(0)/2\delta'(0)$.

- ▶ however in this case t is not a holomorphic operator, because \tilde{T}_a has dimensions $(2 + \delta(n), \delta(n))$
- ▶ b is defined within the logCFT at $c = 0$, but the physical quantities $c'(0)$ and $\delta'(0)$ are not

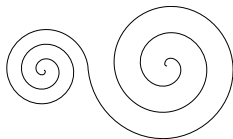
Example II. The $O(n \rightarrow 0)$ model and self-avoiding walks

- ▶ the $O(n)$ field theory with integer n is based on a multiplet of fields (ϕ_1, \dots, ϕ_n) and action

$$S = \int \left[\sum_{a=1}^n (\partial\phi_a)^2 + m_0^2 \sum_{a=1}^n \phi_a^2 + \lambda_0 \left(\sum_{a=1}^n \phi_a^2 \right)^2 \right] d^d r$$

- ▶ on a lattice, the free theory represents a sum over loops, weighted by $e^{-m_0^2(\text{length})}$ and with a factor n for each loop. In this way the model already makes sense for non-integer n .
- ▶ the interaction λ_0 provides a repulsion between different loops and different parts of the same loop: as $\lambda_0 \rightarrow \infty$ we get an ensemble of self-avoiding loops. For m_0^2 large, these are all small, at a critical value the typical size diverges and we have a CFT in the scaling limit.
- ▶ if we compute $\langle \phi_a(r_1)\phi_a(r_2) \rangle$ we get the weighted number of open walks from r_1 to r_2 : at $n = 0$ these are SAWs.

Smirnov's observable ($d = 2$)



- ▶ Smirnov instead considered the quantity $\langle \psi_a(z_0) \psi_a(z) \rangle$ in which the open walks from z_0 to z are also weighted by $e^{-is\theta_{z_0z}}$ where θ_{z_0z} is the winding angle
- ▶ at the critical point and for the correct value of s ($= h_{2,1}(n)$ in the Kac table!), this is a *discretely holomorphic* function of z , i.e. obeys a discrete version of the Cauchy-Riemann relations
- ▶ if this continues to hold in the scaling limit, then
 - a) this can be used to prove that the curves are SLE_κ
 - b) we get a holomorphic conformal field (parafermion) $\psi_a(z)$ with dimensions $(h_{2,1}(n), 0)$

Consider now the OPE

$$\psi_a(z_1) \cdot \psi_b(z_2) = z_{12}^{-2h_{2,1}(n)} \left(\delta_{ab} + \frac{2h_{2,1}(n)}{c(n)} z_{12}^2 \delta_{ab} T(z_1) + \dots \right)$$

- ▶ as $n \rightarrow 0$ there is a potential catastrophe, which is avoided in this case by the collision of T with another operator
- ▶ since ψ_a is a Kac $(2, 1)$ operator, this can only be in the list allowed by the fusion rules, and the only candidate is a Kac $(3, 1)$ operator, since $\lim_{n \rightarrow 0} h_{3,1}(n) = 2$
- ▶ this can be identified physically as the deformation of the operator $T_{ab} \propto (\partial_z \phi_a)(\partial_z \phi_b)$ in the free theory: the trace deforms to T and the traceless part to \tilde{T}_{ab}
- ▶ we can then identify $t \propto T_{11}$, and, since we know both $c(n)$ and $h_{3,1}(n)$, we can compute $b = -c'(0)/2h'_{3,1}(0) = \frac{5}{6}$
- ▶ the correlators $\langle tt \rangle$ and $\langle tT \rangle$ can be physically interpreted in terms of SA loops

Summary

- ▶ we can understand the appearance of logarithmic behaviour in some CFTs which are limits of regular CFTs
- ▶ this involves the presence of a global symmetry under which the operators transform irreducibly in general: however these representations become singular in the limit: logarithms then appear due to collisions of irreducible operators and cancellations in the OPE
- ▶ comparison with a non-interacting limit allows the physical identification of these operators
- ▶ the $c \rightarrow 0$ catastrophe may be avoided in various ways: when this happens due to the collision of another operator with T there is a logarithmic partner t (which however is not necessarily holomorphic in 2d)
- ▶ the parameter $b = -\frac{1}{2}(dc/d\Delta)$ where Δ is the dimension of the operator which collides with T

- ▶ there are important physical quantities (like $c'(0)$) which are not in the logCFT
- ▶ most of this still holds in general dimension d