Conformal Field Theory and Disordered Systems

Logarithmic behaviour at RG fixed points

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- Example 1: random bond ferromagnets

- Example 2: percolation and the spin quantum Hall transition

Expected behaviour at an ordinary critical fixed point.

Any real local density $\Phi(r)$ may always be expanded in a series of scaling operators $\Phi(r) = \sum_i a_i \phi_i(r)$, so that its two-point correlation function takes the form

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{A_{ij}}{r^{x_i+x_j}}$$

- Scaling dimensions x_i: eigenvalues of the dilatation operator $\mathcal{D}$ (usually discrete).

- Conformal invariance: off-diagonal terms $x_i \neq x_j \Rightarrow A_{ij} = 0$.

- Reflexion positivity: $A_{ij} \geq 0$.

$\Rightarrow$ only pure powers (or sums of them) should appear. [NB logs can arise from marginally irrelevant operators, but these are of form $(1 + g \ln r)^{\gamma}$ – vanish *at* the fixed point.]

• Relax positivity (e.g. on using replica trick for a disordered system) – then it is possible that

$$\frac{A_1}{r^{2x_1}} + \frac{A_2}{r^{2x_2}} \sim \frac{\ln r}{r^{2x_1}}$$

if $x_1 \to x_2$ with $A_1 \sim -A_2 \to \infty$ such that $A_1(x_1 - x_2)$ finite.

•This happens generically as $n \to 0$ in the replica trick and for other similar problems:

- Percolation ($Q \to 1$ limit of Potts model)
- Self-avoiding walks (polymers – $n \to 0$ limit of O(n) model)
- ...

[NB Such logs have already been shown to be possible in some 2d CFT's (e.g. with SUSY). The $n \to 0$ limit shows *how* they arise.]

Example: random-bond Ising model for $2 < d < 4$.

$$H = -\sum_{rr'}(J+\delta J_{rr'})s(r)s(r') \sim H^0 + \int \delta J(r)E(r)d^dr$$

$E(r)$ = local energy density; $\overline{\delta J(r)\delta J(r')} = g\delta_{rr'}$.

Introduce replicas $a = 1,\ldots,n$ and average

$$H_R \equiv \sum_a H_a^0 - g\int \sum_{a\neq b} E_a(r)E_b(r)d^dr$$

$[g] = y_g = d - 2x_E^0$ – relevant if $y_g > 0 \Rightarrow$ Harris criterion $d\nu^0 < 2$.

$\Rightarrow$ RG flow to random fixed point; perturbation theory valid if $y_g \ll 1$. For *any* scaling operator ϕ

$$x_\phi = x_\phi^0 + (2b_\phi/b)y_g + O(y_g^2)$$

where b and b_ϕ are the coefficients in the OPEs $\Phi\cdot\Phi = 1 + b\Phi + \cdots$ and $\phi\cdot\Phi = b_\phi\phi + \cdots$, and $\Phi = \sum_{a\neq b} E_a E_b$.

• Irrespective of validity of PT, the scaling operators ϕ should be arranged into *irreducible* representations of whatever symmetries are present.

E.g. $\phi \sim$ energy density $E(r) \sim \sum_{r'} s(r)s(r')$. Replica permutation symmetry: $\{E_a\}$ reducible, because $\sum_a E_a$ transforms according to identity rep. So define

$$\begin{aligned} E &\equiv \sum_a E_a \\ \widetilde{E}_a &\equiv E_a - (1/n)\sum_b E_b \end{aligned}$$

Then

$$\begin{aligned} x_E(n) &= x_E^0 + \frac{1}{2}(1-n)y_g + O(y_g^2) \\ x_{\widetilde{E}}(n) &= x_E^0 + \frac{1}{2}y_g + O(y_g^2) \end{aligned}$$

$$
\begin{aligned}
\langle E(r)E(0)\rangle &= n\left(\langle E_1E_1\rangle + (n-1)\langle E_1E_2\rangle\right) \\
&\sim nA(n)r^{-2x_E(n)} \\
\langle \widetilde{E}_a(r)\widetilde{E}_a(0)\rangle &= (1-1/n)\left(\langle E_1E_1\rangle - \langle E_1E_2\rangle\right) \\
&\sim (1-1/n)\widetilde{A}(n)r^{-2x_{\widetilde{E}}(n)}
\end{aligned}
$$

For consistency $A(0) = \widetilde{A}(0)$. Subtracting and letting $n \to 0$,

$$
\begin{aligned}
\overline{\langle E(r)\rangle\langle E(0)\rangle} &= \lim_{n\to 0}\langle E_1(r)E_2(r)\rangle \\
&\sim 2A(0)x'_E(0)\ln r/r^{-2x_E(0)}
\end{aligned}
$$

NB *connected* energy-energy correlation function ($\to$ specific heat) is $\overline{\langle EE\rangle} - \overline{\langle E\rangle\langle E\rangle} = \langle E_1E_1\rangle - \langle E_1E_2\rangle$ and contains no log.

$$
\frac{\overline{\langle E(r)\rangle\langle E(0)\rangle}}{\overline{\langle EE\rangle_c}} \sim 2x'_E(0)\ln r
$$

with a *universal* coefficient.

NB Such a phenomenon does not occur for the correlation functions of the local magnetisation $s(r)$, for the RBIM, because $s_a(r)$ transforms *irreducibly* under larger symmetry

$$[\text{replica permutations}] \otimes [s_a \to -s_a]^n$$

so that $x_\sigma(n) = x_{\tilde{\sigma}}(n)$ for all n.

But...

Logarithms are ubiquitous in all higher correlation functions!

In any CFT (ie, any RG fixed point with rotational symmetry and short-range interactions), the form of the OPE of any scaling operator with itself is

$$\phi(r)\cdot\phi(0) = \frac{a_\phi}{r^{2x_\phi}}(1 + \frac{x_\phi}{c} r^d T + \cdots) \qquad (*)$$

where $T =$ *stress tensor* , scaling dimension d.

[Follows from $\langle T\phi\phi\rangle \sim x_\phi$ and $\langle TT\rangle \sim c/r^{2d}$ – all indices have been suppressed.]

c (central charge in $d = 2$) *vanishes* in any theory with partition function $Z = 1$ – eg, replicas $n \to 0$, $Q \to 1$, ...

- Obvious problem in (*) as $c \to 0$!

$$\phi(r)\cdot\phi(0) = \frac{a_\phi}{r^{2x_\phi}}(\mathbf{1} + \frac{x_\phi}{c} r^d T + \cdots) \qquad (*)$$

Two resolutions:

- $a_\phi \to 0$ as $c \to 0$ - this is what happens in percolation as $Q \to 1$ – all connected correlation functions vanish $\propto (Q-1)$, to get percolation connectedness correlations need to take $\partial/\partial Q|_{Q=1}$.

- (generic solution) there are other operators omitted from (*) which collide with T as $c \to 0$, and cancel the singularity:

$$\phi(r)\cdot\phi(0) = \frac{a_\phi}{r^{2x_\phi}}(\mathbf{1} + \frac{x_\phi}{c} r^d T + \frac{b}{c} r^{x_{\widetilde{T}}} \widetilde{T} + \cdots)$$

This is what happens in the RBIM:

$$T \equiv \sum_a T_a \qquad \text{the true stress tensor }, x_T = d$$

$$\widetilde{T}_a \equiv T_a - (1/n)\sum_b T_b, \qquad x_{\widetilde{T}} \to d \text{ only as } n \to 0$$

Example:

$$\overline{\langle s(r_1)s(r_2)\rangle\langle s(r_3)s(r_4)\rangle} - \overline{\langle s(r_1)s(r_2)\rangle} \cdot \overline{\langle s(r_3)s(r_4)\rangle}$$

$$\sim \text{angle} - \text{dependent factor } \times \left|\frac{r_{12}r_{34}}{r_{13}r_{24}}\right|^d \ln\left|\frac{r_{12}r_{34}}{r_{13}r_{24}}\right|$$

for $|r_{12}r_{34}/r_{13}r_{24}| \ll 1$.

[Such effects, while rather general, are obviously small].

• Are there scaling operators whose dimension $x \to 0$ as $c \to 0$, so they become degenerate with 1, giving logarithmic effects at leading order?

Percolation and the random cluster model

Q-state Potts model. $s(r) \in \{1, 2, 3, \ldots, Q\}$.

$$
\begin{aligned}
Z &= \mathrm{Tr}\, e^{J \sum_{rr'} \delta_{s(r),s(r')}} \\
&\propto \mathrm{Tr} \prod_{rr'} (1 + x\delta_{s(r),s(r')}) \\
&= \sum_{\text{cluster configs}} x^{\text{no. of bonds}} Q^{\text{no. of clusters}}
\end{aligned}
$$

• Given a rectangular region $L \times W$, on average how many distinct clusters $\langle N_c \rangle$ cross the sample? (Figure). [NB, for $L \gg W$ we expect this to be $\propto L$, and $\propto L/W$ at $p = p_c$.

• Maps onto the mean conductance in a network model of the spin *quantum* Hall transition [Gruzberg, Ludwig and Read, PRL **82**, 4254 (1999)].

– Electrons move on the links of a directed L-lattice (Figure).

– At each node they scatter into each of the two possible channels via a given S-matrix.

– On each link the amplitude gets multiplied by a *random* SU(2) matrix.

– Conductance $g \propto \mathrm{Tr}\, \sum_{ij} |G_{ij}|^2$, where $G_{ij} =$ sum over all paths $i \to j$, each weighted by a product of S-matrix elements and SU(2) matrices along the path.

– Quenched average over SU(2) matrices: only paths which are *hulls* of percolation clusters which cross the sample survive $\Rightarrow g \propto \langle N_c \rangle$.

Boundary conditions on upper and lower edges:

- Free (f): Potts spins unconstrained
- Fixed (1): Boundary spins fixed into state 1.

$$Z_{ff} = \langle Q^{N_c+N_1+N_2+N_b} \rangle \qquad Z_{11} = \langle Q^{N_b} \rangle$$
$$Z_{1f} = \langle Q^{N_2+N_b} \rangle \qquad Z_{f1} = \langle Q^{N_1+N_b} \rangle$$

so that

$$\langle N_c \rangle = (\partial/\partial Q)|_{Q=1}(Z_{ff}Z_{11}/Z_{f1}Z_{1f})$$

Periodic boundary conditions in L-direction:

$$Z_{ij} = \mathrm{Tr}\, \mathbf{T}_{ij}^L \sim e^{\pi((c/24)-x_{ij})(L/W)}$$

where x_{ij} are boundary scaling dimensions. For $L \ll W$ only the smallest counts: if $i = j$ this is $x = 0$.

So

$$\langle N_c \rangle \sim (\partial/\partial Q)|_{Q=1} e^{2\pi x_{1f}(Q)(L/W)}$$

NB, if $\langle N_c \rangle \propto L/W$, it must be that $x_{1f}(Q = 1) = 0$! (and the coefficient is $\propto (\partial/\partial Q)|_{Q=1} x_{1f}(Q)$.

In fact, $x_{1f}(Q)$ is known for $Q = 2, 3$ to be $x_{1,2}$ in the Kac classification. $\Rightarrow$

$$\langle N_c \rangle \sim (\sqrt{3}/4)(L/W)$$

Result for finite L/W at $p = p_c$:

$$2\langle N_c \rangle = 1 - \frac{\sqrt{3}}{2\pi} \left(\ln(1-\eta) + 2 \sum_{m=1}^{\infty} \frac{(\frac{1}{3})_m}{(\frac{2}{3})_m} \frac{(1-\eta)^m}{m} \right)$$

where $\eta = (1-k)^2/(1+k)^2$ and $L/W = K(1-k^2)/2K(k^2)$.

• In general, scaling dimensions like $x_{1f}(Q)$, which $\to 0$ as $c \to 0$, correspond to correlation functions which measure *topological* properties of the system.

Example: in a percolation problem, what is the probability $P(M)$ that exactly M clusters must be crossed to connect a given point P to the boundary? (Figure).

$$\sum_M P(M){Q'}^M \sim L^{-x(Q')} \quad \text{(or } \xi^{-x(Q')} \text{for } p \neq p_c\text{)}$$

where there is an explicit formula for $x(Q')$ (and $x(1) = 0$).

• $P(M) \sim P(0)(|\ln(p_c - p)|/2\pi)^{2M}/(2M)!$, for $M \ll \langle M\rangle$, where $P(0) \sim |p_c - p|^{5/36}$, and

• $\langle M\rangle \sim (1/3\sqrt{3}\pi)|\ln(p_c - p)|$.

• Similar results for O(n) model (self-avoiding loops).

⇒ Logarithms everywhere! ⇐

Summary

• Logarithmic factors in correlation functions occur in all quenched random systems (not just in 2d) and in other systems described by an $n \to 0$ limit (self-avoiding walks, percolation,...)

•In the RBIM they appear in disconnected correlation functions, not in linear susceptibilities.

•In percolation they describe topological rather than local properties of clusters (which is why they haven't been seen in most studies).

•In the mapping of the SU(2) spin quantum Hall transition to percolation they are essential in giving a finite conductivity at the critical point (= band centre).

•A similar mechanism must operate in the (so far unknown) CFT which presumably describes the integer quantum Hall transition.