ENTANGLEMENT ENTROPY & QUANTUM FIELD THEORY

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Outline.

- entropy as a measure of entanglement
- entropy and QFT on branched Riemann surfaces
- exact results for 2d CFT and massive QFT
- higher dimensional generalisations

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Quantum Entanglement

- quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$
- complete commuting set of observables: A can observe a subset A, B the complement B
- in general A's measurements are entangled with those of B
- one measure of the amount of entanglement is the *entropy*:
- A's reduced density matrix

$$\rho_A = \mathrm{Tr}_B\, \rho$$

von Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \qquad (= S_B)$$

- other measures of entanglement have been suggested, but more difficult to obtain general analytic results

Simple Example

- 2 spin-$\frac{1}{2}$ degrees of freedom: A observes one, B the other
- if $|\Psi\rangle = \cos\theta|\uparrow\rangle|\downarrow\rangle + \sin\theta|\downarrow\rangle|\uparrow\rangle$

then

$$S_A = -\cos^2\theta\log\cos^2\theta - \sin^2\theta\log\sin^2\theta$$

- $S_A = 0$ for unentangled states, e.g. $|\uparrow\rangle|\downarrow\rangle$
- S_A is maximal ($=\log 2$) for maximally entangled states $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle)$
- `[Bennett et al.]` if supplied with M copies of this state, by purely local operations and classical communication, A can produce $M' < M$ maximally entangled states, with the optimal conversion ratio $M'/M \sim S_A$ as $M \to \infty$

In this talk we consider the case when the quantum system is a (cut-off) quantum field theory in d+1 dimensions, usually in its ground state

- [Srednicki,1993]: for a massless QFT in $d > 1$, $S_A \propto$ (surface area of A) Λ^{d-1}: black hole physics???
- [Holzhey,Larsen,Wilczek,1994]: for a massless $d = 1$ QFT (a conformal field theory), $S_A \sim (c/3)\log(\Lambda\ell)$, where
  - c is the central charge of the CFT
  - ℓ measures the linear dimension of A

Some of the questions to be addressed:

- for $d = 1$, what happens
  - if A consists of several disjoint pieces?
  - if the whole system is itself finite?
  - if the theory is massive?
  - if the whole system is in a mixed state, e.g. corresponding to finite temperature?
- for $d > 1$, what happens if the theory is massive?

von Neumann entropy and Riemann surfaces

- QFT in 1+1 dimension
- let $\{\hat{\phi}(x)\}$ be a set of fundamental fields $\{\hat{\phi}(x)\}$, their eigenvalues and corresponding eigenstates be $\{\phi(x)\}$ and $\otimes_x|\{\phi(x)\}\rangle$ respectively.
- at finite temperature β^{-1}, density matrix is

$$\rho(\{\phi(x'')''\}|\{\phi(x')'\}) = Z(\beta)^{-1}\langle\{\phi(x'')''\}|e^{-\beta\hat{H}}|\{\phi(x')'\}\rangle\,,$$

where $Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}}$ is the partition function.

- path integral representation:

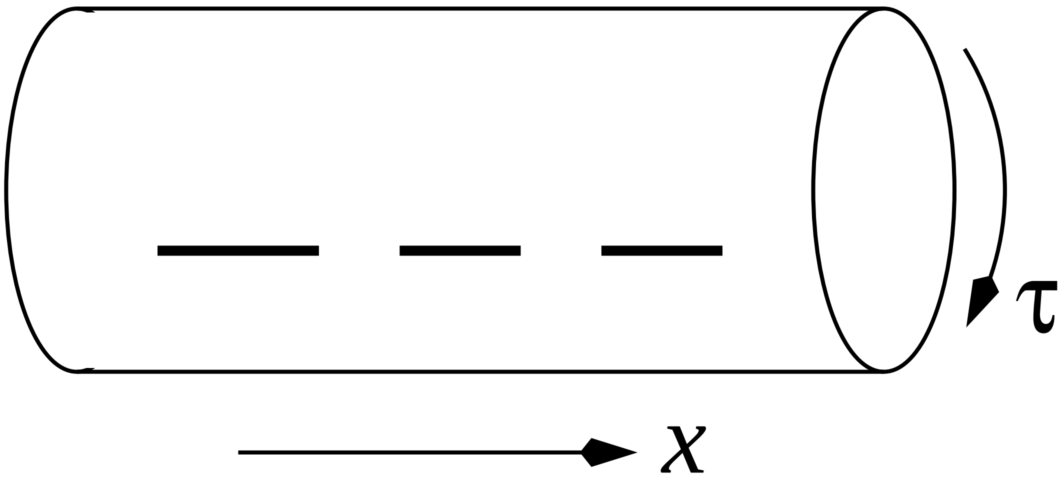
$$\rho = \frac{1}{Z}\int[d\phi(x,\tau)]\prod_x \delta(\phi(x,0)-\phi(x')')\delta(\phi(x,\beta)-\phi(x'')'')e^{-S_E}$$

where $S_E = \int_0^\beta L_E d\tau$, with L_E the euclidean lagrangian.

- let $(u_1, v_1), (u_2, v_2), \ldots$ be a set of disjoint intervals of the line and let $A = \{\phi(x); x \in \cup_j(u_j, v_j)\}$
- reduced density matrix

$$\rho_A = \int \prod_{x\in B} d[\phi(x,0)]\delta(\phi(x,\beta) - \phi(x,0))\,\rho$$

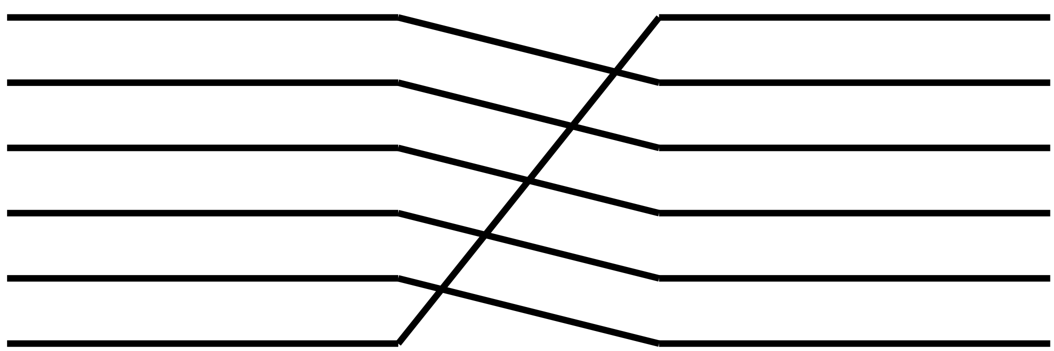
$\Rightarrow$ sew together the edges along $\tau = 0, \beta$, leaving slits open along each interval (u_j, v_j) $\Leftarrow$



Now use

$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A = -\left.\frac{\partial}{\partial n}\right|_{n=1} \mathrm{Tr}\,\rho_A^n$$

- for integer $n \geq 1$ the rhs is given by n copies of the path integral for S_A, sewn together cyclically along the slits: an n-sheeted Riemann surface



- this has a unique continuation to $\mathrm{Re}\, n \geq 1$, and its derivative at $n = 1$ gives S_A:

$$S_A = -\left.\frac{\partial}{\partial n}\right|_{n=1} \left(\frac{Z_n}{Z_1^n}\right)$$

where Z_n is the (unnormalised) partition function on the n-sheeted surface

- in this ratio, the worst of the UV divergences in $\ln Z_n$ ($\propto \Lambda^2$) cancel, leaving only log divergences
- as $\beta \to \infty$, all IR divergences cancel

CFT on n-sheeted Riemann surface

- branch points are orbifold points in string theory whose target space is $(\mathrm{CFT})^n$
- for $c = 1$, need to compute $\det(-\nabla^2)$ on the Riemann surface
- instead, consider the expectation value of the stress tensor $\langle T \rangle$

Simplest example: a single slit (u, v)

- conformal mapping $w \to z = ((w-u)/(w-v))^{1/n}$ uniformises the n-sheeted surface

$$T(w) = (dz/dw)^2\, T(z) + \tfrac{c}{12}\{z, w\}\,,$$

where $\{z, w\}$ = Schwartzian derivative $(z'''z' - \frac{3}{2}z''^2)/z'^2$.
$\langle T(z) \rangle_{\mathbf{C}} = 0 \Rightarrow$

$$\langle T(w) \rangle = \frac{c(1-(1/n)^2)}{24} \frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

- compare with the form of the correlator of T with 2 primary fields (conformal Ward identity):

$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle} = \frac{\Delta_\Phi (v-u)^2}{(w-u)^2(w-v)^2}$$

- conformal WI completely determines transformation properties under conformal mappings $w \to w' = w + \alpha(w)$ which are the same on each sheet, by insertion of

$$(1/2\pi i)\int T(w)\alpha(w)dw + \text{cc.}$$

Z_n/Z_1^n transforms in the same way as $\langle\Phi_n(u)\Phi_{-n}(v)\rangle^n$, where Φ has scaling dimension $\Delta_\Phi = c(1-(1/n)^2)/24$

In the plane,

$$\frac{Z_n}{Z_1^n} = c_n\left(\frac{a}{v-u}\right)^{(c/6)(n-1/n)}$$

where $c_1 = 1$, so

$$S_A = (c/3)\log\left((v-u)/a\right) + c_1'$$

Results for related geometries

Under a conformal mapping $w \to f(w)$

$\langle\Phi(w_1)\Phi(w_2)\rangle = |f'(w_1)|^{2\Delta}|f'(w_2)|^{2\Delta}\langle\Phi(f(w_1))\Phi(f(w_2))\rangle$

and same is true for Z_n/Z_1^n

- e.g. $f(w) = (\beta/2\pi)\log w$ maps plane to a cylinder with $0 \leq \tau < \beta$
- $\Rightarrow$ result at finite temperature:

$$S_A = (c/3)\log\left((\beta/\pi a)\sinh(\pi\ell/\beta)\right) + c_1'$$

where $\ell = v - u$.

- for $\ell >> \beta$, $S_A \sim \pi c\ell/3\beta$ `[Blöte,JC,Nightingale ;Affleck;1986]`

– or, at zero temperature, but in a finite system of length L with periodic bc:

$$S_A = (c/3)\log\left((L/\pi a)\sin(\pi\ell/L)\right) + c_1'$$

Similar results for open boundaries:

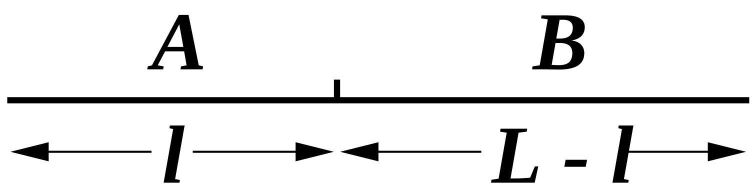
- semi-infinite system $x \geq 0$, slit from 0 to v:

$$Z_n/Z_1^n \propto \langle \Phi_n(v) \rangle^n \sim (2v)^{-2\Delta}$$

so

$$S_A \sim (c/6)\ln(v/a)$$

- conformally map this to a finite system of length L, divided into an interval A of length ℓ and B of length $L - \ell$:



$$S_A = (c/6)\log\left((L/\pi a)\sin(\pi\ell/L)\right) + 2g + c_1'$$

(g = boundary entropy.)

General case

- A is the union of N intervals (u_j, v_j)
- uniformising transformation has the form

$$z = \prod_k (w - w_k)^{\alpha_k}$$

where $\Sigma_k \alpha_k = 0$, $w_k = u_j$ or v_j.

- compute $\langle T(w) \rangle$ from Schwartzian derivative
- under what conditions does this have the form of conformal Ward identity $\langle T(w) \, \Pi_k \, \Phi_k(w_k) \rangle$?
- theorem: this happens iff $\alpha_k = \pm\alpha$ for each k:
- this is precisely the case we need, with $\alpha = 1/n$

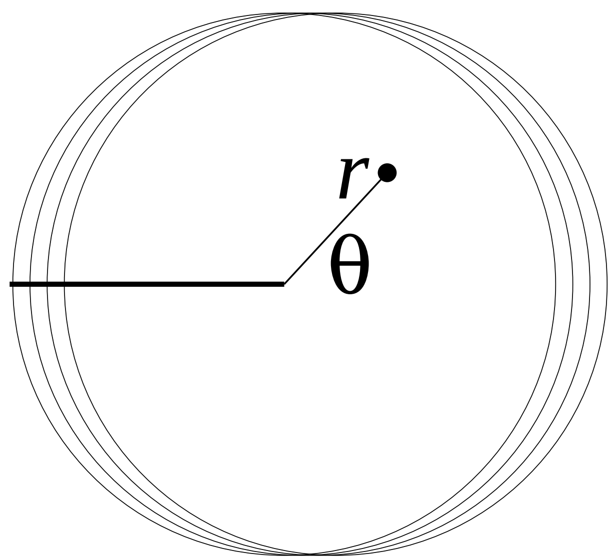
so

$$\mathrm{Tr}\, \rho_A^n \sim c_n^N \left(\frac{\Pi_{j<k}(u_k - u_j)(v_k - v_j)}{\Pi_{j \le k}(v_k - u_j)} \right)^{(c/6)(n-1/n)}$$

from which S_A follows by differentiation wrt n.

Massive field theory in 1+1 dimensions

- explicit calculations e.g. for quantum spin chains show that $S_A \to$ a finite limit as $\ell_A \to \infty$, if the correlation length ξ ($\propto$ (mass $m)^{-1}$) is finite
- how does this limit behave as $m \to 0$?
- use simplest geometry: infinite system with $A = (0, \infty)$, $B = (-\infty, 0)$
- Riemann surface consists of n planes sewn together along negative real axis



- use properties of stress tensor:
- for $n = 1$, $\langle T \rangle = \langle T_{zz} \rangle = 0$, but $\langle \Theta \rangle = \langle T^\mu_\mu \rangle \neq 0$, reflecting breaking of scale invariance
- for $n > 1$

$$\begin{aligned} \langle T(z, \bar{z}) \rangle &= F_n(z\bar{z})/z^2 \\ \langle \Theta(z, \bar{z}) \rangle - \langle \Theta \rangle_1 &= G_n(z\bar{z})/(z\bar{z}) \end{aligned}$$

Conservation equation

$$\partial_{\bar z} T + \tfrac{1}{4}\partial_z \Theta = 0 \qquad \Rightarrow$$

$$(z\bar z)\left(F_n' + \tfrac{1}{4}G_n'\right) = \tfrac{1}{4}G_n$$

- expect F_n and $G_n \to 0$ exponentially fast for $|z| \gg m^{-1}$, while as $|z| \to 0$, $F_n \to (c/24)(1-n^{-2})$, $G_n \to 0$

$$\int \left(\langle\Theta\rangle_n - \langle\Theta\rangle_1\right) d^2R = -\pi n(c/6)(1-n^{-2})$$

But the lhs is just $-(\partial/\partial m)(\log Z_n - n\log Z_1)$, so

$$\frac{Z_n}{Z_1^n} = c_n (ma)^{(c/12)(n-1/n)}$$

and finally

$$S_A \sim -(c/6)\log(ma)$$

- if A consists of a number of intervals each of length $\gg m^{-1}$, we expect above result to be multiplied by the number of boundary points between A and B

Example - massive free field theory

$$\mathcal{S} = \int \tfrac{1}{2}\left((\partial_\mu\varphi)^2 + m^2\varphi^2\right) d^2r$$

Use

$$\frac{\partial}{\partial m^2}\log Z_n = -\tfrac{1}{2}\int G_n(\mathbf{r},\mathbf{r})d^2r$$

where $G_n(\mathbf{r},\mathbf{r}')$ is the two-point correlation function in the n-sheeted geometry:

$$G_n(\mathbf{r},\mathbf{r}) = \sum_{k=0}^{\infty} d_k I_{k/n}(mr)K_{k/n}(mr)$$

- sum over k is divergent, but formally we get

$$\int d^dr\, G(r) = \sum_k d_k \int I_{k/n}(mr)K_{k/n}(mr)r\,dr = \frac{1}{2nm^2}\sum_k d_k k$$

- interpreting the sum as $2\zeta(-1) = -\frac{1}{6}$, we agree with the general result with $c = 1$
- can be done properly by cutting off sum over k: UV divergence then cancels in the correct combination $G_n - nG_1$ which gives $(\partial/\partial m^2)\log(Z_n/Z_1^n)$
- result also verified for lattice models solvable by Baxter's corner transfer matrix method: in this case complete spectrum of ρ_A is known, even for $\xi \sim a$
- free-field case can also be used to analyse effects of finite system size

Higher dimensions

- example: massive free field theory with A = half-space $x > 0$, $d-1$ other dimensions $x_\perp$
- Fourier transform wrt $x_\perp \to$ effective mass $m^2 + k_\perp^2$

$$S_A = -\frac{1}{12}\int d^{d-1}r_\perp \int \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}} \log \frac{k_\perp^2 + m^2}{k_\perp^2 + \Lambda^2}$$

- $S_A \propto$ surface area $\mathcal{A} = \int d^{d-1}r_\perp$ `[Srednicki]`
- coefficient UV divergent $\propto \Lambda^{d-1}$
- but there is a finite non-leading term $\propto m^{d-1}$
- conjecture (a simple generalisation of ordinary scaling for the free energy):

for any massive QFT in $d > 1$, the leading *singular* term in $S_A/\mathcal{A}$ goes like m^{d-1} with a *universal* coefficient

Summary

- von Neumann entropy is an analytically tractable measure of entanglement
- for 1d systems whose continuum limit is a 1+1-dimensional relativistic QFT its form can be computed in many cases
- this is universal and proportional to c, which counts the number of massless UV degrees of freedom
- intimately related to stress tensor
- higher dimensions: universal behaviour?
- are other measures of entanglement also related to c?