SCHRAMM-LOEWNER EVOLUTION CALOGERO-SUTHERLAND MODELS and CFT

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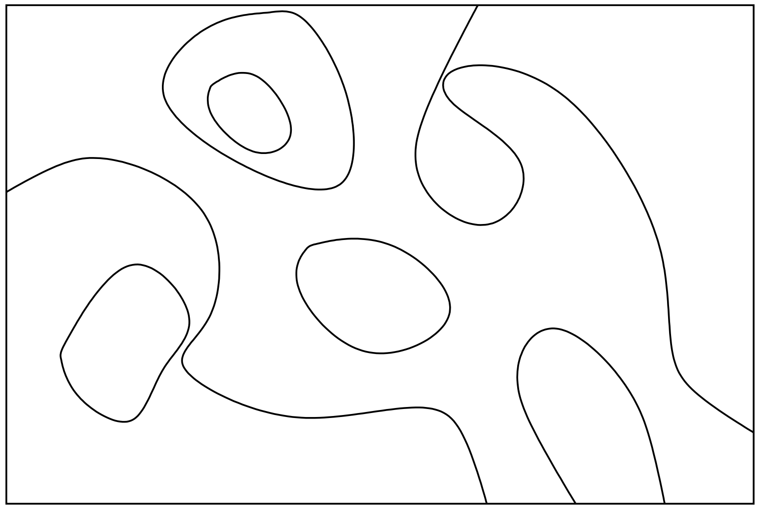
Outline.

- brief introduction to SLE
- radial SLE
- multiple SLEs and Dyson's process
- realization of Dyson's ensemble in 2D critical systems
- relation to CFT and C-S model

Some of this work is in `math-ph/0301039`
These overheads available at
`http://www-thphys.physics.ox.ac.uk/users/JohnCardy/home.html`

2D critical systems as loop gases



Many 2D critical systems can be realised as gases of random self-avoiding non-intersecting loops and open curves, eg:

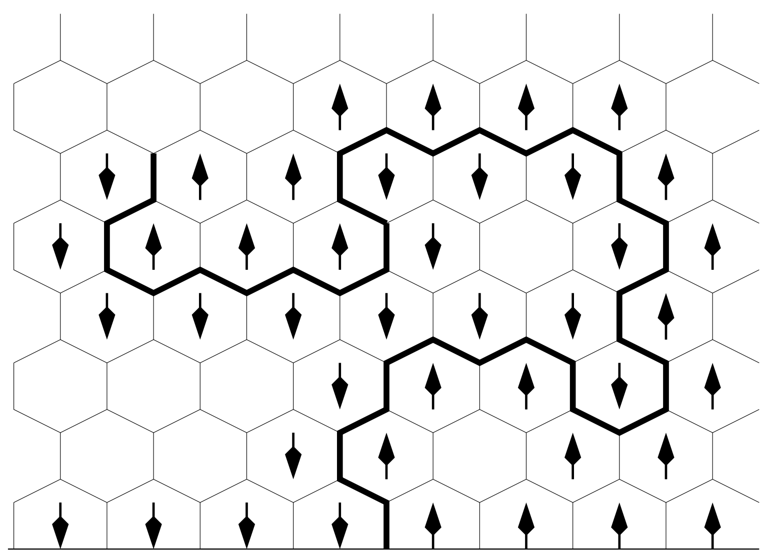
- Q-state Potts model: boundaries of Fortuin-Kasteleyn clusters
  - $Q = 1$: boundaries of percolation clusters
- 'high-temperature' graphs of $O(n)$ model
  - $n = 2$: level lines of a random surface at the roughening transition
  - $n = 1$: (dual to) boundaries of critical Ising spin clusters
  - $n = 0$: self-avoiding random walks

Schramm (Stochastic) Loewner Evolution (SLE)

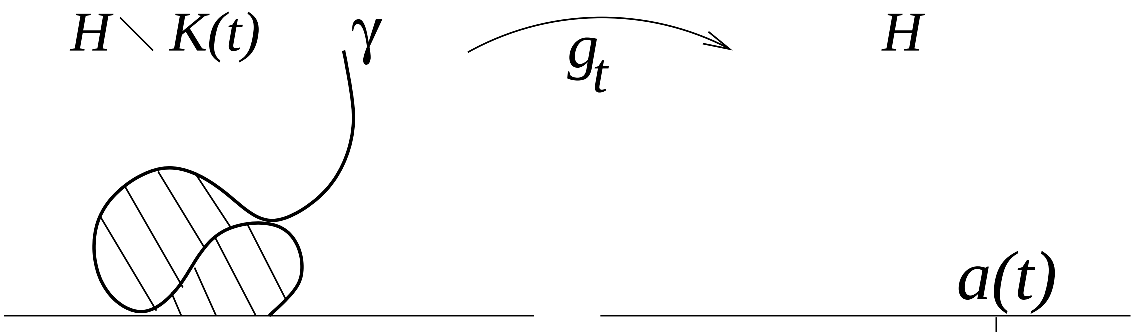
`[Lawler, Schramm, Werner]`

- SLE is a theory of the *continuum limit* of random curves in 2D lattice models
- it assumes and exploits *conformal invariance*
- so far, proved in only a few cases, but if limit exists and is conformally invariant, it is given by SLE.

SLE in the half plane (chordal SLE)



- example: Ising domain wall conditioned by the boundary conditions to start at 0 and end at ∞.
- exploration process: ensemble of non-crossing paths can be generated dynamically
- SLE describes the continuum limit of this process



- let K_t be the 'hull' of the curve up to time t (= the curve ∪ any regions enclosed by curve)
- [Riemann] there is a conformal mapping $g_t : H \setminus K_t \to H$, normalized such that

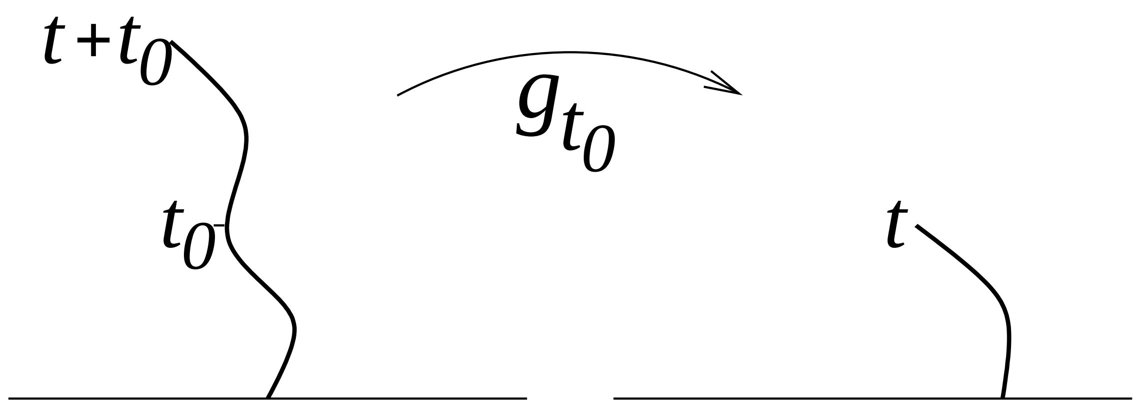
$$g_t(z) \sim z + \frac{2t}{z} + \cdots, \quad \text{as} \quad z \to \infty$$

- [Loewner] evolution of $g_t(z)$ satisfies

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a(t)}$$

with $a(t)$ a real continuous function which characterizes K_t.

- [Schramm] if the law of the path is conformally invariant, then $a(t)$ must be a 1D Brownian motion



- law of g_{t+t_0} same as law of g_t
- scale invariance: $z \to \lambda z$, $t \to \lambda^2 t$
  $\Rightarrow$ law of $\lambda^{-1} a(\lambda^2 t)$ same as law of $a(t)$.
- only solution is Brownian motion: $\dot{a}(t) = \eta(t)$, with $\langle \eta(t')\eta(t'') \rangle = \kappa \delta(t' - t'')$

$\Rightarrow$ Problems in 2D critical systems reduced to problems in 1D Brownian motion

Some simple properties of SLE:

- fractal dimension of the trace is $1 + \kappa/8$ $(\kappa \leq 8)$
- for $\kappa \leq 4$ it is a simple curve and does not touch the real axis (almost surely)
- for $\kappa > 4$ it repeatedly touches itself and the real axis: every point in $\mathbf{H}$ eventually is swallowed by the hull K_t
- different values of κ correspond to different universality classes of 2D critical behavior:
- Q-state Potts model: boundaries of FK clusters $\sqrt{Q} = -2\cos\frac{4\pi}{\kappa}$, with $4 \leq \kappa \leq 8$
  - boundaries of percolation clusters $\kappa = 6$
- 'high-temperature' graphs of $O(n)$ model, $n = -2\cos\frac{4\pi}{\kappa}$, with $2 \leq \kappa \leq 4$
  - $n = 2$: level lines of a random surface $\kappa = 4$
  - $n = 1$: boundaries of Ising spin clusters $\kappa = 3$
  - $n = 0$: self-avoiding random walks $\kappa = \frac{8}{3}$
- central charge of corresponding CFT

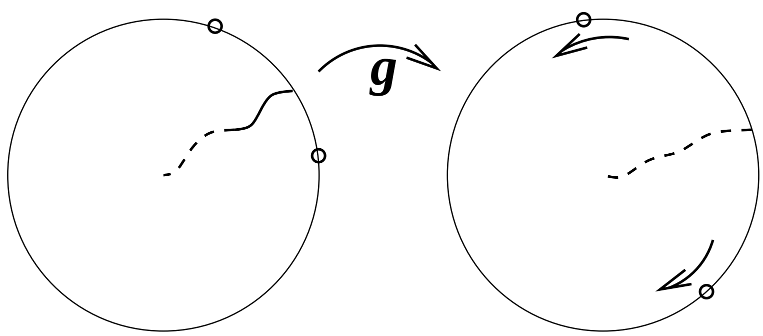
$$c = 1 - 6\left(\frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}\right)^2$$

Radial SLE

- path in unit disc $\mathbf{U}$, starting at $z = e^{i\theta}$ and conditioned to end at $z = 0$
- $g_t(z)$ conformally maps $\mathbf{U} \setminus K_t$ to $\mathbf{U}$, such that $g_t(0) = 0$, $g_t'(0) > 0$
- radial Loewner equation

$$\frac{dg_t(z)}{dt} = -g_t(z)\,\frac{g_t(z) + e^{i\theta + ia(t)}}{g_t(z) - e^{i\theta + ia(t)}}$$

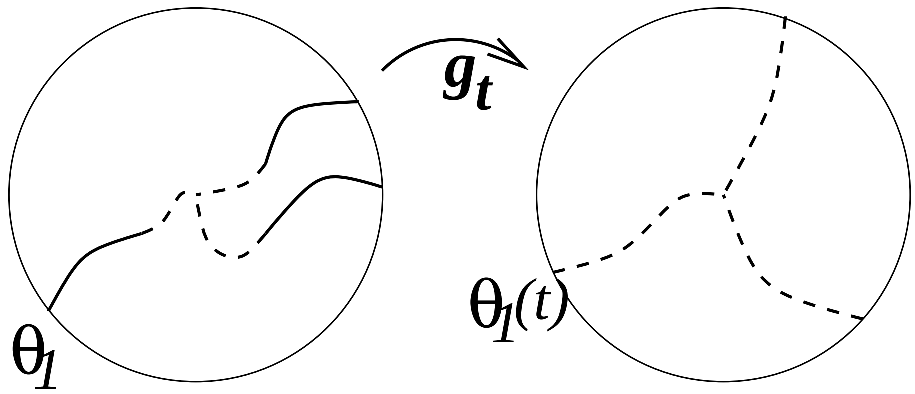
- stochastic version: take $a(t) = \sqrt{\kappa}B(t)$.
- g_t takes the growing tip γ_t into $e^{i\theta(t)}$ where $\theta(t) = \theta(0) + \sqrt{\kappa}B(t)$; other points on the boundary are repelled from this point



- infinitesimal transformation:

$$\begin{aligned} \dot{\theta} &= \cot((\theta - \theta(t))/2) \\ \dot{\theta}(t) &= \eta(t) \end{aligned}$$

Multiple radial SLEs



- N nonintersecting paths in unit disc from $\{e^{i\theta_j}\}$ to 0 (assume $\kappa \leq 4$)
- G_t conformally maps $\mathbf{U} \setminus \cup_j K_j(t)$ to $\mathbf{U}$
- generate this by infinitesimally advancing each path in turn:

$$G_{t+dt} \circ G_t^{-1} = g_{dt}^{(N)}(\theta_N(t)) \circ \cdots \circ g_{dt}^{(1)}(\theta_1(t))$$

Then

$$\frac{dG_t(z)}{dt} = -\frac{G_t(z)}{N} \sum_{j=1}^{N} \frac{G_t(z) + e^{i\theta_j(t)}}{G_t(z) - e^{i\theta_j(t)}}$$

where

$$\dot{\theta}_j(t) = \sum_{k \neq j} \cot(\theta_j(t) - \theta_k(t))/2 + \eta_j(t)$$

- this is Dyson's process

Dyson's process and random matrices

$$\dot{\theta}_j(t) = -\frac{\partial V}{\partial \theta_j} + \eta_j(t)$$

with $V = -\Sigma_{j<k} \ln \sin^2(\theta_j - \theta_k)/2$.

- tends to equilibrium distribution

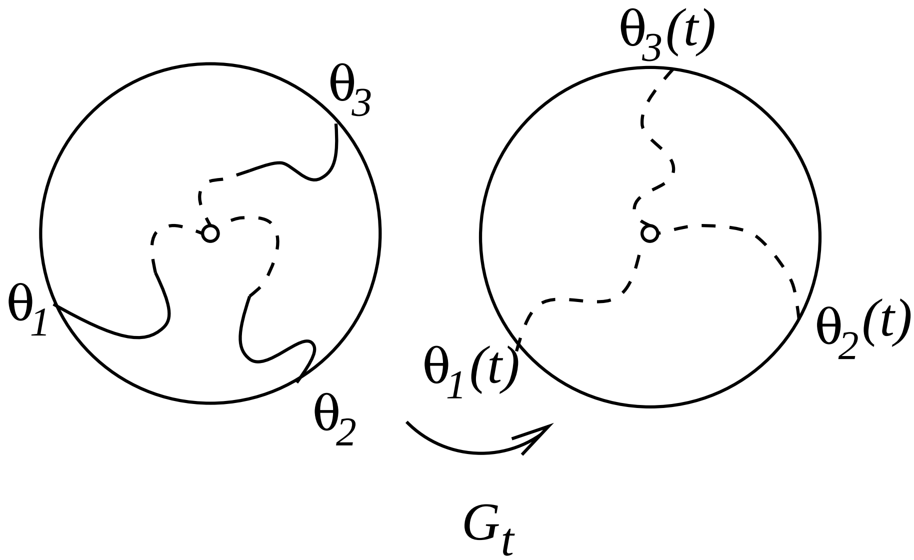
$$P_{\text{eq}}(\theta_1, \ldots, \theta_N) \propto e^{-V/(\kappa/2)} \propto \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^\beta$$

with

$$\beta = 4/\kappa$$

- P_{eq} is joint distribution function for eigenvalues of $N \times N$ unitary matrix:
  - $\beta = 2$ (unitary)
  - $\beta = 1$ (orthogonal)
  - $\beta = 4$ (symplectic)
- but note that other values of β are allowed here

What does this mean?



- suppose there are exactly N curves starting on $r = \epsilon$ and ending on $Re^{i\theta_j}$
- conformal invariance $\Rightarrow$

$$\langle \mathcal{O} \rangle_{\{\theta_j\}} = \Big\langle \langle \mathcal{O} \rangle_{\{\theta_j(t)\}} \Big\rangle_{\{\eta_j(t');0<t'<t\}}$$

$\Rightarrow$ for $R \gg \epsilon$ the joint distribution of the $\{\theta_j\}$ is given by $P_{\rm eq}$.

Comparison with conformal field theory

- in O(n) conformal field theory the object in question corresponds to a correlation function

$$\langle \underbrace{\phi_{a_1}(e^{i\theta_1})\ldots\phi_{a_N}(e^{i\theta_N})}_{\text{boundary operators}}\ \underbrace{\Phi_{a_1,\ldots,a_N}(0)}_{\text{bulk operator}}\rangle$$

with O(n) labels a_j all unequal. In operator language,

$$\underbrace{\langle\theta_1,\ldots,\theta_N|}_{\text{boundary state}}(\epsilon/R)^{L_0+\overline{L}_0}\ \underbrace{|\Phi\rangle}_{\text{highest wt state}}$$

- infinitesimal G_t is

$$z\to z-\delta t\frac{z}{N}\sum_j\frac{z+e^{i\theta_j}}{z-e^{i\theta_j}}=z+\delta t(z+\sum_{n>0}a_nz^{n+1})$$

and similarly for $\bar z$

- generator is $G=L_0+\overline{L}_0+\Sigma_{n>0}(a_nL_n+a_n^*\overline{L}_n)$

$$G|\text{hws}\rangle=(L_0+\overline{L}_0)|\text{hws}\rangle=(\Delta+\overline{\Delta})|\text{hws}\rangle$$

- on the other hand G acting to the left just moves the points around the boundary:

$$\langle\theta_1,\ldots,\theta_N|G=\langle\theta_1',\ldots,\theta_N'|$$

- decompose CFT Hilbert space into the finite-dimensional space spanned by the boundary states $|\theta_1, \ldots, \theta_N\rangle$ and its complement
- in this basis G has form

$$G = \begin{pmatrix} \mathcal{L}^\dagger & 0 \\ \star & \star \end{pmatrix}$$

so

$$G|\text{hws}\rangle = G \begin{pmatrix} |B\rangle \\ \star \end{pmatrix} = \begin{pmatrix} \mathcal{L}^\dagger |B\rangle \\ \star \end{pmatrix} = (\Delta + \overline{\Delta}) \begin{pmatrix} |B\rangle \\ \star \end{pmatrix}$$

so

$$\mathcal{L}^\dagger |B\rangle = (\Delta + \overline{\Delta})|B\rangle$$

$\Rightarrow$ eigenvalues of $\mathcal{L}^\dagger$ are highest weights of the CFT

What is $\mathcal{L}$?

- the generator for the Fokker-Planck eqn $\dot{P} = \mathcal{L}P$

$$\mathcal{L} = \sum_{j=1}^{N} \left(\frac{\partial}{\partial\theta_j} \frac{\partial V}{\partial\theta_j} + \frac{\kappa}{2} \frac{\partial^2}{\partial\theta_j^2} \right)$$

- $\mathcal{L} P_{\text{eq}} = 0$
- $\mathcal{L}^\dagger \neq \mathcal{L}$, but if

$$\mathcal{H} \equiv P_{\text{eq}}^{-1/2} \mathcal{L} P_{\text{eq}}^{1/2}$$

then $\mathcal{H}^\dagger = \mathcal{H}$ and $\mathcal{H} P_{\text{eq}}^{1/2} = 0$.

What is $\mathcal{H}$?

$$\mathcal{H} = -\frac{\kappa}{2}\sum_j \frac{\partial^2}{\partial\theta_j^2} + \frac{2-\kappa}{2\kappa}\sum_{j<k}\frac{1}{\sin^2(\theta_j-\theta_k)/2} - \text{const.}$$

– the Calogero-Sutherland hamiltonian

- ground state of $\mathcal{H}_{CS}$ is $P_{\rm eq}^{1/2}$
- ground state energy in N-particle sector gives bulk N-leg exponent
- ground state energy with modified boundary conditions gives other bulk exponents (eg one-arm exponent (LSW))
- all the other excited state energies of $\mathcal{H}_{CS}$ give *new* highest weight exponents of the O(n) CFT

Summary and Some Further Questions

- Dyson's ensemble is realized in 2D critical systems for values of $\beta = 4/\kappa$ other than canonical ones – eg $\beta = \frac{4}{3}$ (Ising model) $\beta = \frac{3}{2}$ (self-avoiding walks)
- a connection between the (integrable) C-S hamiltonian and conformal field theory: eigenvalues of $\mathcal{H}_{CS}$ are highest weights of the CFT
- dualities?
- spin C-S generalisations?
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