SLE – A NEW WAY OF THINKING ABOUT CFT

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These overheads available at
`http://www-thphys.physics.ox.ac.uk/users/JohnCardy/durham.ps`

- CFT: **C**onformal **F**ield **T**heory
- SLE: **S**chramm (or **S**tochastic) **L**oewner **E**volution

Outline.

- brief review of CFT
- random curves in lattice models
- SLE describes the continuum limit of these curves
- SLE and CFT
- multiple SLEs and Calogero-Sutherland models
- SLEs on higher genus

Review of $2d$ CFT

CFT: massless $2d$ ($1+1d$) relativistic QFT

- *string theory:* $2d$ world sheet of the string
  - degs. of freedom $\Leftrightarrow$ target space
  - CFT on higher genus $\Leftrightarrow$ string loop expansion
  - boundary CFT $\Leftrightarrow$ open strings
  - scaling dimensions $\Leftrightarrow$ spectrum of string
  - correlation functions $\Leftrightarrow$ S-matrix
- *statistical mechanics:* continuum limit of critical
  - classical systems in $2d$, or
  - quantum systems in $1+1d$
  - degs. of freedom $\Leftrightarrow$ observables

Scale invariance: massless field theories are 'invariant' under scale transformations

- non-trivial because of renormalisation:
- scaling operators $\phi(\mathbf{r}) \to \lambda^x \phi(\lambda\mathbf{r})$

scale invariance + locality $\Rightarrow$ 'conformal invariance'

- in $2d$, conformal mappings $\equiv$ analytic functions $z \to f(z)$

$$\phi(z,\bar{z}) \to |f'(z)|^x\,\phi(f(z),\overline{f(z)})$$

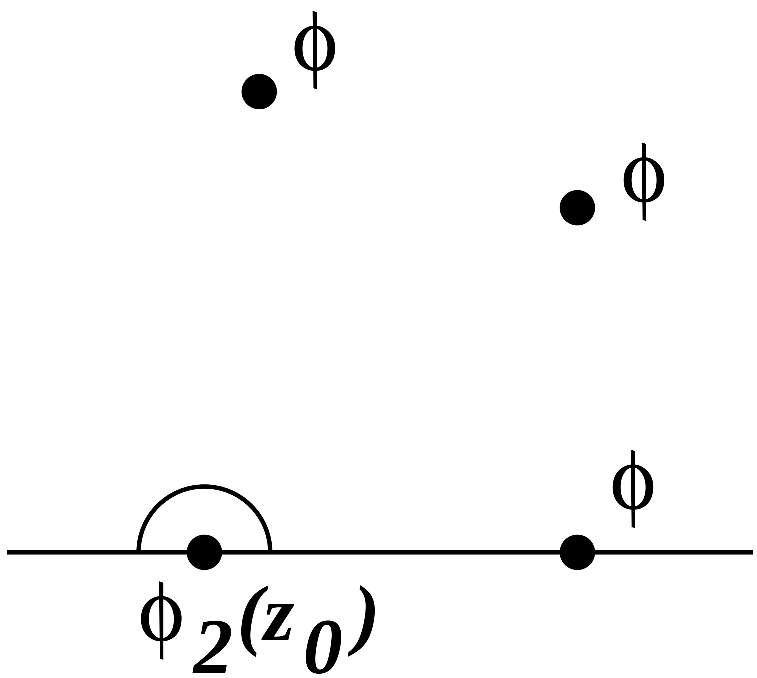
or, infinitesimally with $f(z) = 1 + \alpha(z)$:

$$\phi(z_0) \to \phi(z_0) + \tfrac{1}{2}x\alpha'(z_0)\phi + \alpha(z_0)\partial_z\phi + \cdots$$

- what if $\alpha(z)$ is singular at z_0?
  eg $\alpha(z) = \Sigma_n a_n (z - z_0)^{n+1}$
- $\phi \to \phi + \Sigma_n a_n L_n \phi$
- $L_0 \phi = \frac{1}{2} x \phi$ and $L_{-1}\phi = \partial_z \phi$, but $L_{-2}\phi$, $L_{-3}\phi, \ldots$ are *new* scaling operators
- they give a highest wt representation of the Virasoro algebra
- null vectors:
  - eg $L_{-2}\phi_2 \propto L_{-1}^2 \phi_2 = \partial_z^2 \phi_2$
  - ϕ_2 operators are basic building blocks, through OPE fusion rules: eg

$$\phi_2(z_1) \cdot \phi_2(z_2) \sim 1 + \phi_3(z_1)$$

Correlation functions of degenerate operators satisfy linear PDEs



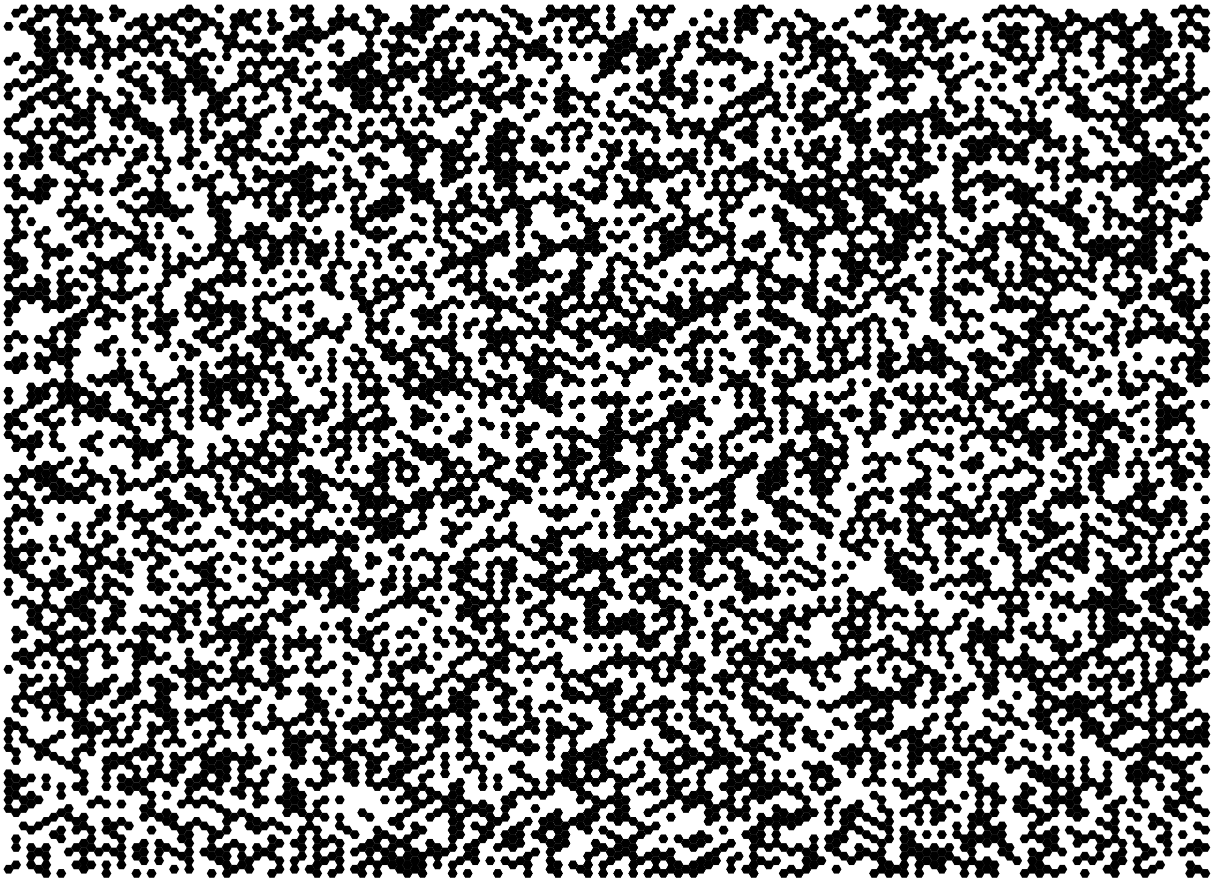
$$z \to z + \frac{a_{-2}}{z - z_0}$$

$$\frac{\partial^2}{\partial z_0^2}\langle \phi_2(z_0)\phi(z_1)\ldots\rangle \propto$$
$$\left(\frac{x/2}{(z_1 - z_0)^2} + \frac{1}{z_1 - z_0}\frac{\partial}{\partial z_1} + \cdots\right)\langle \phi_2(z_0)\phi(z_1)\ldots\rangle$$

2d critical behaviour

- continuum limit of lattice models (Ising, Potts,..) believed to be a CFT.
- 'Local' operators (magnetisation,...) $\Rightarrow$ scaling operators $\phi(z)$
- in many cases, these are degenerate $\Rightarrow$ scaling dimensions x (critical exponents) and correlation functions.

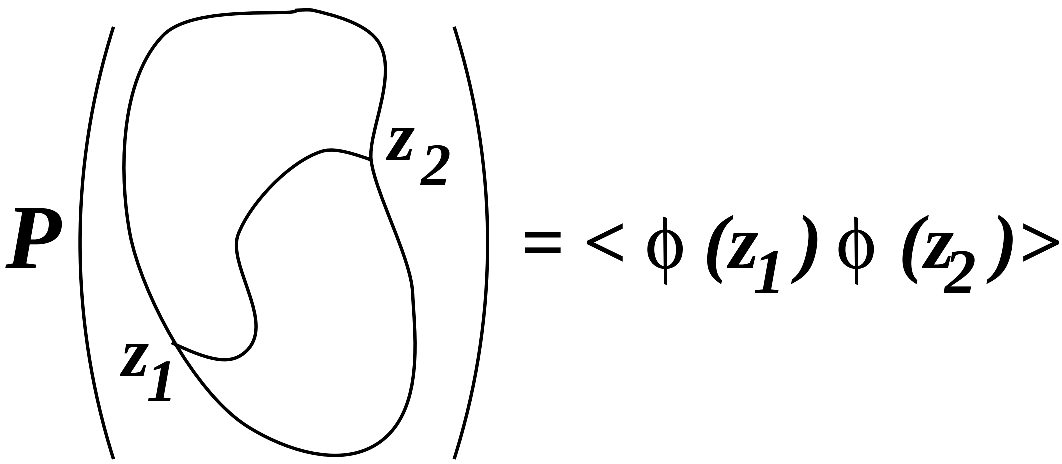
2D critical systems as loop gases



Many 2D critical systems can be realised as gases of random self-avoiding non-intersecting loops and open curves, either as:

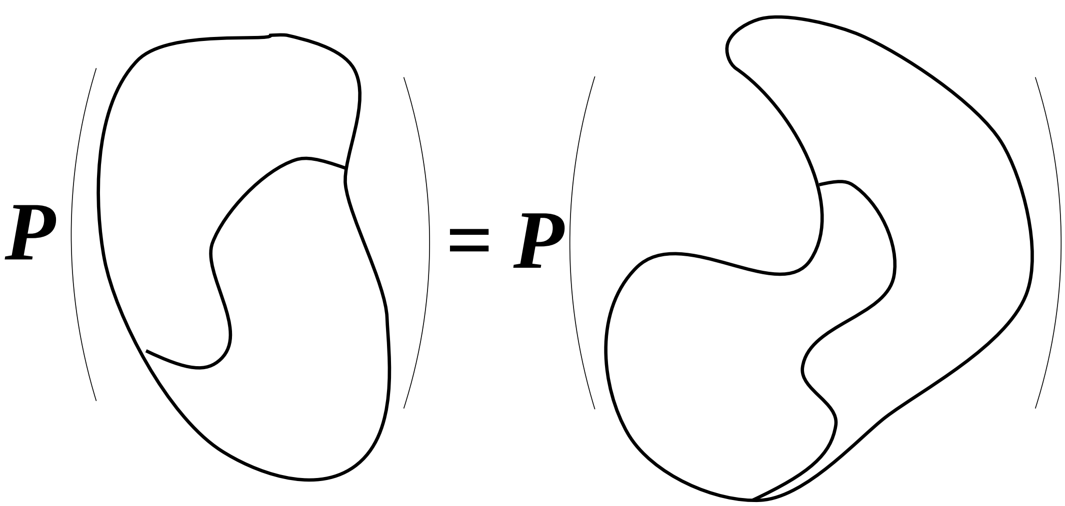
- boundaries of clusters (percolation, Ising), or
- boundaries of dual clusters (XY model), or
- directly (self-avoiding walks)

- intersections with boundary can be thought of a local (boundary) scaling operators:



- conjecture (1984): ϕ is degenerate at level 2 $\Rightarrow$ its correlators satisfy 2nd order PDEs

At criticality, in continuum limit, the measure on these curves is conformally **invariant**

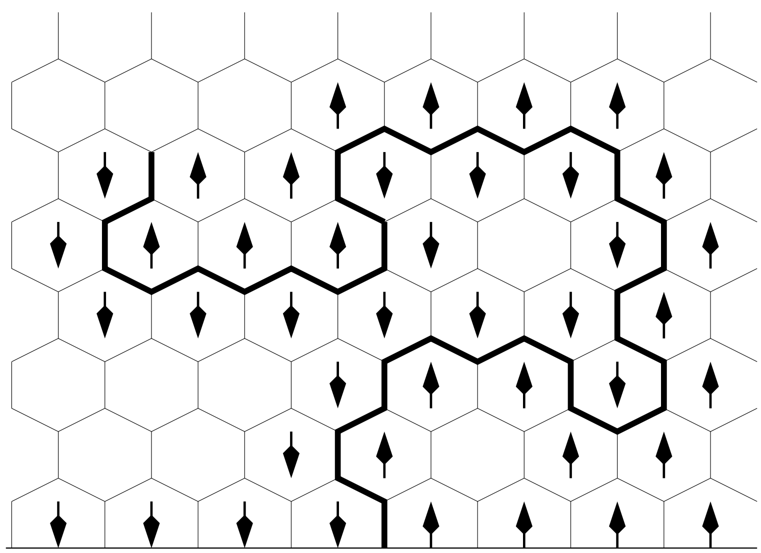


Schramm (Stochastic) Loewner Evolution (SLE)

[Lawler, Schramm, Werner]

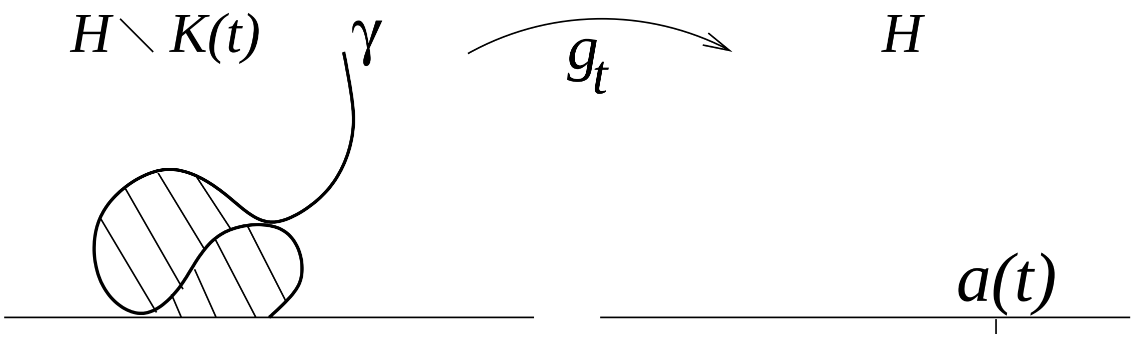
- SLE is a theory of the *continuum limit* of random curves in 2D lattice models
- it assumes and exploits *conformal invariance*

SLE in the half plane (chordal SLE)



- example: Ising domain wall conditioned by the boundary conditions to start at 0 and end at ∞.
- exploration process: ensemble of non-crossing paths can be generated dynamically

Continuum limit of this process



- let K_t be the 'hull' of the curve up to time t (= the curve $\cup$ any regions enclosed by curve)
- `[Riemann]` there is a conformal mapping $g_t : H \setminus K_t \to H$, normalized such that

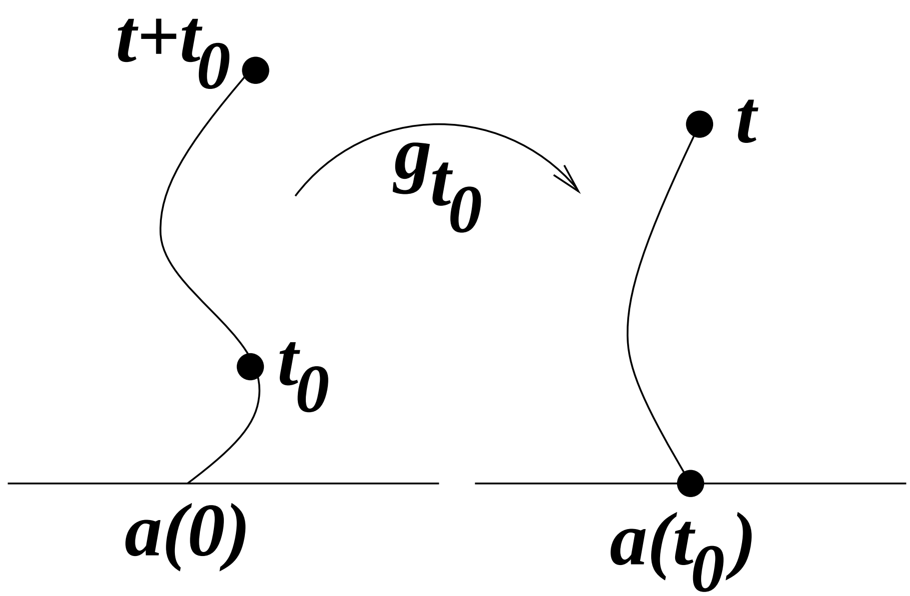
$$g_t(z) \sim z + \frac{2t}{z} + \cdots, \quad \text{as} \quad z \to \infty$$

- `[Loewner]` evolution of $g_t(z)$ satisfies

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a(t)}$$

with $a(t)$ a real continuous function which characterizes K_t.

• [Schramm] if the law of the path is conformally invariant, then $a(t)$ must be a 1D Brownian motion



• law of g_{t+t_0} same as law of g_t

• scale invariance: $z \to \lambda z$, $t \to \lambda^2 t$
$\Rightarrow$ law of $\lambda^{-1} a(\lambda^2 t)$ same as law of $a(t)$.

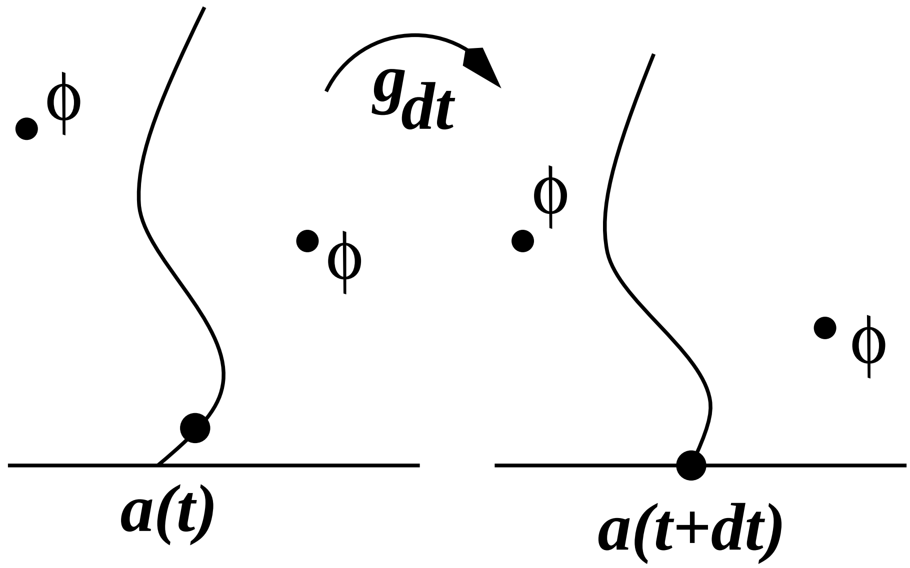
• only solution is Brownian motion:
$\dot{a}(t) = \eta(t)$, with $\langle \eta(t')\eta(t'')\rangle = \kappa\delta(t'-t'')$

$\Rightarrow$ Problems in 2D critical systems reduced to problems in 1D Brownian motion

Some simple properties of SLE:

- fractal dimension of the trace is $1 + \kappa/8$ ($\kappa \leq 8$)
- for $\kappa \leq 4$ it is a simple curve and does not touch the real axis (almost surely)
- for $\kappa > 4$ it repeatedly touches itself and the real axis: every point in $\mathbf{H}$ eventually is swallowed by the hull K_t
- different values of κ correspond to different universality classes of 2D critical behavior:
  - boundaries of Ising clusters $\kappa = 3$
  - boundaries of percolation clusters $\kappa = 6$
  - Kosterlitz-Thouless transition $\kappa = 4$
  - self-avoiding walks $\kappa = \frac{8}{3}$

Differential equations



- consider condtional expectation value $C = \langle \phi(z_1)\phi(z_2)\ldots\rangle_{a(0)}$ given that there is a curve starting at $a(0)$
- map this under

$$g_{\delta t}(z) \approx z + \frac{2\delta t}{z - a(0)}$$

- $\phi(z_j)$ transform as under L_{-2}
- $a(\delta t) \approx a(0) + \int_0^{\delta t} \eta(t')dt'$
- $\langle C(a(\delta t))\rangle_\eta \approx C(a(0)) + \frac{1}{2}\kappa\delta t\, \partial_a^2 C$

$\Rightarrow$ same PDE as if we had assumed a ϕ_2 operator at $a(0)$ $\Leftarrow$

$\mathbf{SLE}_\kappa \equiv$ CFT with central charge

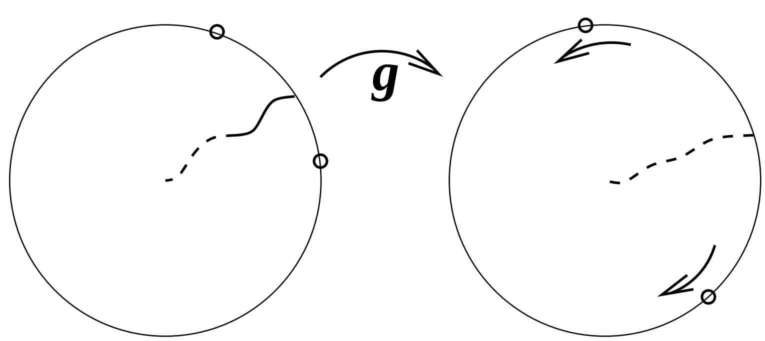
$$c = 1 - 6\left(\frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}\right)^2$$

Radial SLE

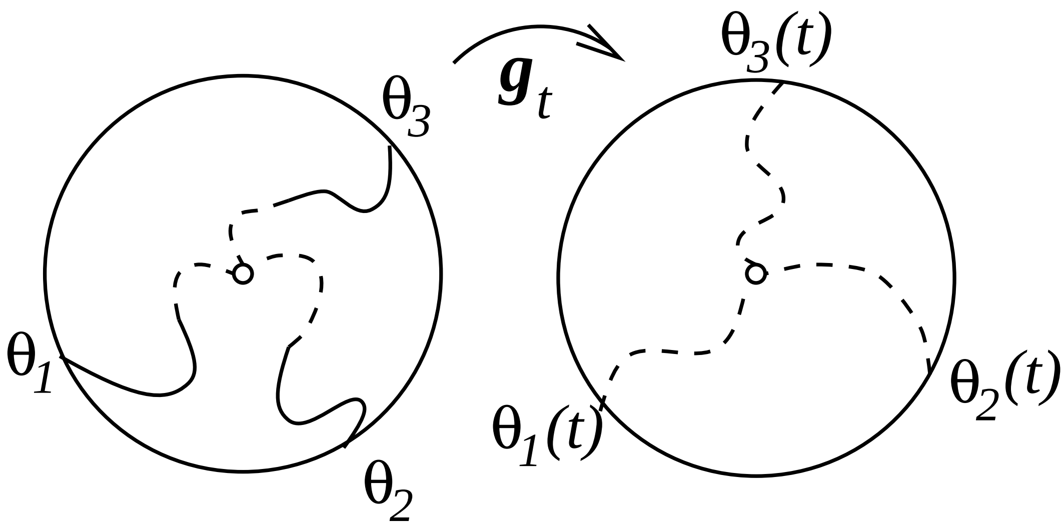
- path in unit disc $\mathbf{U}$, starting at $z = e^{i\theta}$ and conditioned to end at $z = 0$
- $g_t(z)$ conformally maps $\mathbf{U} \setminus K_t$ to $\mathbf{U}$, such that $g_t(0) = 0$, $g_t'(0) > 0$
- radial Loewner equation

$$\frac{dg_t(z)}{dt} = -g_t(z)\,\frac{g_t(z) + e^{i\theta(t)}}{g_t(z) - e^{i\theta(t)}}$$

- stochastic version: take $\theta(t)$ to be Brownian motion



Multiple radial SLEs



- N nonintersecting paths in unit disc from $\{e^{i\theta_j}\}$ to 0 (assume $\kappa \leq 4$)
- g_t conformally maps $\mathbf{U} \setminus \cup_j K_j(t)$ to $\mathbf{U}$
- generate this by infinitesimally advancing each path in turn:

Then

$$\frac{dg_t(z)}{dt} = -\frac{g_t(z)}{N} \sum_{j=1}^{N} \frac{g_t(z) + e^{i\theta_j(t)}}{g_t(z) - e^{i\theta_j(t)}}$$

where

$$\dot{\theta}_j(t) = \sum_{k \neq j} \cot(\theta_j(t) - \theta_k(t))/2 + \eta_j(t)$$

- this is Dyson's process for eigenvalues of random unitary matrices

Comparison with conformal field theory

- in CFT the object in question corresponds to a correlation function

$$\langle \underbrace{\phi_2(e^{i\theta_1}) \ldots \phi_2(e^{i\theta_N})}_{\text{boundary operators}} \quad \underbrace{\Phi(0)}_{\text{bulk operator}} \quad \rangle$$

- near the origin, $g_t(z) \sim e^t\, z$ acts like a scale transformation on $\Phi(0)$
- one the unit circle, $g_t(z)$ acts as a 2nd order differential operator on the points $(\theta_1, \ldots, \theta_N)$:

$$\mathcal{D} \propto \sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + \cdots$$

In fact

$$\mathcal{D} = S^{-1}\, \mathcal{H}\, S$$

with

$$\mathcal{H} = -\frac{\kappa}{2}\sum_j \frac{\partial^2}{\partial\theta_j^2} + \frac{2-\kappa}{2\kappa}\sum_{j<k}\frac{1}{\sin^2(\theta_j-\theta_k)/2} - \text{const.}$$

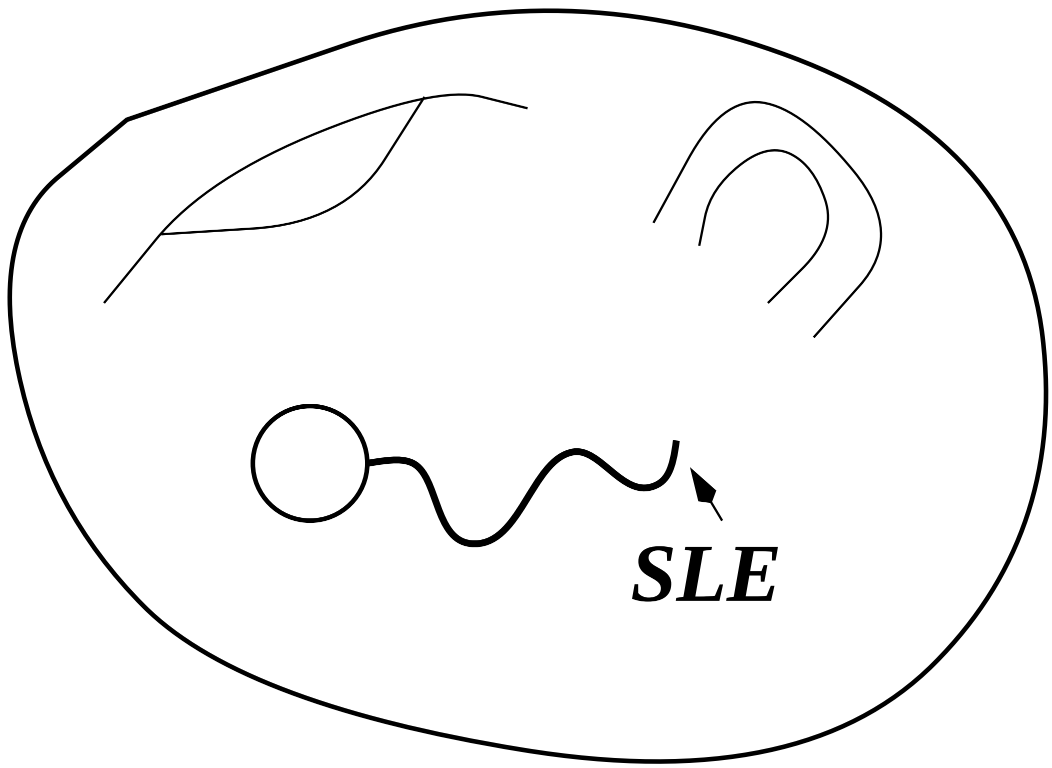
– the Calogero-Sutherland hamiltonian

- left- and right-eigenvalues of $\mathcal{D}$ are the same as those of $\mathcal{H}$ $\Rightarrow$

scaling dimensions of bulk primary operators Φ which couple to N boundary ϕ_2 operators are given by the spectrum of Calogero-Sutherland hamiltonian $\mathcal{H}$

- since all other degenerate operators are given by fusion of ϕ_2s, true for correlators of these as well.

SLE on higher genus



- as curve grows, the *moduli* change.
- SLE $\Rightarrow$ random exploration of moduli space
- averaging over Brownian motions $\Leftrightarrow$ integrating over moduli, as in higher loop string theory

Summary

- SLE gives a new way of thinking about CFT
- conformal **invariance** is built in from the start
- geometro-probabilistic *vs.* algebraic
- some results that are hard to understand or derive from one point of view are easy in the other (and *vice versa*)
- e.g. the relevance of the Calogero-Sutherland hamiltonian to
  - boundary ⇒ bulk CFT
  - open string spectrum ⇒ closed string spectrum
- SLE on higher genus explores moduli space
- relation to Dyson's process and random matrices

Further selected reading:

- Mathematical reviews:
  - W. Werner, math.PR/0303354
  - G. Lawler, www.math.cornell.edu/∼lawler/book.ps
- Relation to CFT
  - M. Bauer and D. Bernard, hep-th/0210015 and other more recent articles
- Multiple SLEs and Calogero-Sutherland
  - J. Cardy, math-ph/0301039, hep-th/0310291