Self-avoiding Walks, Branched Polymers, Confinement, Supersymmetry and Dimensional Reduction

John Cardy
Theoretical Physics and All Souls College,
Oxford

- JC, cond-mat/0107223, J. Phys. A **34**, L665 (2001).

- C. Richard, A. J. Guttmann and I. Jensen, cond-mt/0107329; J. Phys. A **34**, L495 (2001).

- D. Brydges and J. Imbrie, math-phy/0107005.

Outline

- Statistical mechanics problem: self-avoiding loops in the plane, weighted by their area;

- Field theory formulation: supersymmetric Abelian Higgs model;

- Exact RG beta-function;

- Confinement: branched polymers;

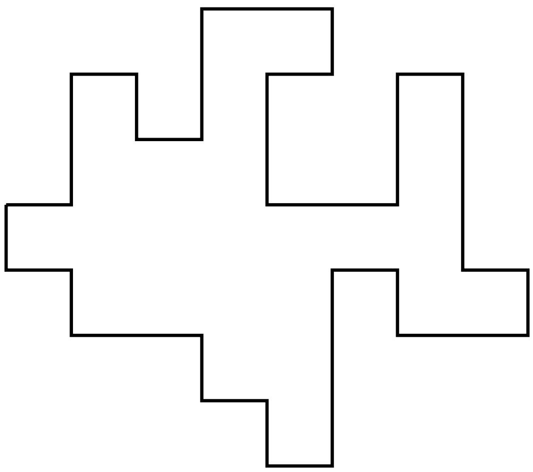
- Branched polymers = "Supersymmetric gas";

- Exact beta-function;

- Dimensional reduction $\Rightarrow$ exact result for $d = 2$;

- Back to self-avoiding loops.

• Stat. mech. problem: counting planar polygons on a lattice weighted by perimeter N, and area A:



Generating function

$$G^{(r)}(x,p) = \sum_{N,A} m^{(r)}_{N,A}\, x^N\, e^{-pA}$$

$p = 2d$ pressure.

• when $p = 0$, $\sum_A m^{(r)}_{N,A} \sim N^{\alpha-2}\mu^N$, so $G^{(r)}(x,0) \sim (x_c - x)^{1-\alpha}$, with $x_c = \mu^{-1}$, lattice-dependent; α universal.

- $(x, p) = (x_c, 0)$ is a multicritical point: *scaling function*

$$G^{(r)}_{\text{sing}} = p^{\beta} \, F((x_c - x)p^{-\gamma})$$

- when $p = 0$, $G^{(r)} \sim (x_c - x)^{1-\alpha} \Rightarrow F(u) \sim u^{\beta/\gamma}$ with $\beta/\gamma = 1 - \alpha$;

- $\langle A \rangle \sim \langle R^2 \rangle \sim \langle N \rangle^{2\nu} \sim (x_c - x)^{-2\nu} \Rightarrow \gamma = 1/2\nu$.

- expect $F(u) = b_1 \tilde{F}(b_2 u)$ where $\tilde{F}(\cdot)$ is universal.

Exact result:

$$\widetilde{F}(u) = \frac{\mathsf{Ai}'(u)}{\mathsf{Ai}(u)}$$

where

$$\mathsf{Ai}(u) \propto \int_{-\infty}^{\infty} e^{iut+it^3/3} dt$$

Moreover

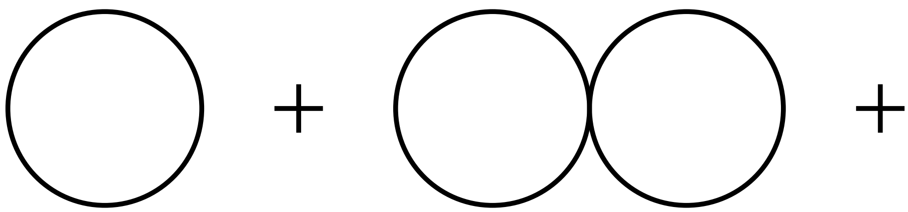
$$\gamma = \frac{2}{3}, \quad \beta = \frac{1}{3} \quad \Rightarrow \quad \alpha = \tfrac{1}{2}, \quad \nu = \tfrac{3}{4}$$

Field theory formulation

• Loops = vacuum diagrams for (complex) scalar field theory

$$S = \int [|\nabla\phi|^2 + m_0^2|\phi|^2 + \lambda|\phi|^4] d^2r$$

$m_0^2 - m_{0c}^2 \sim x_c - x$; $\lambda > 0 \leftrightarrow$ self-repulsion.



Cancel unwanted diagrams by introducing scalar fermionic partner: $|\phi|^2 \to \phi^*\phi + \psi^*\psi$. $\to$ Global SUSY.

$G^{(r)}(x, p = 0) \sim \langle\phi^*\phi\rangle$.

• couple to area by thinking of this as a Wilson loop: in $2d$ coupling to a gauge field gives exact area law = linear confining potential.

$$\begin{aligned} \nabla_\mu &\longrightarrow \nabla_\mu - igA_\mu \\ S &\longrightarrow S + \int F_{\mu\nu}^2 d^2r \end{aligned}$$

- SUSY Abelian Higgs model.

- $p \leftrightarrow g^2$

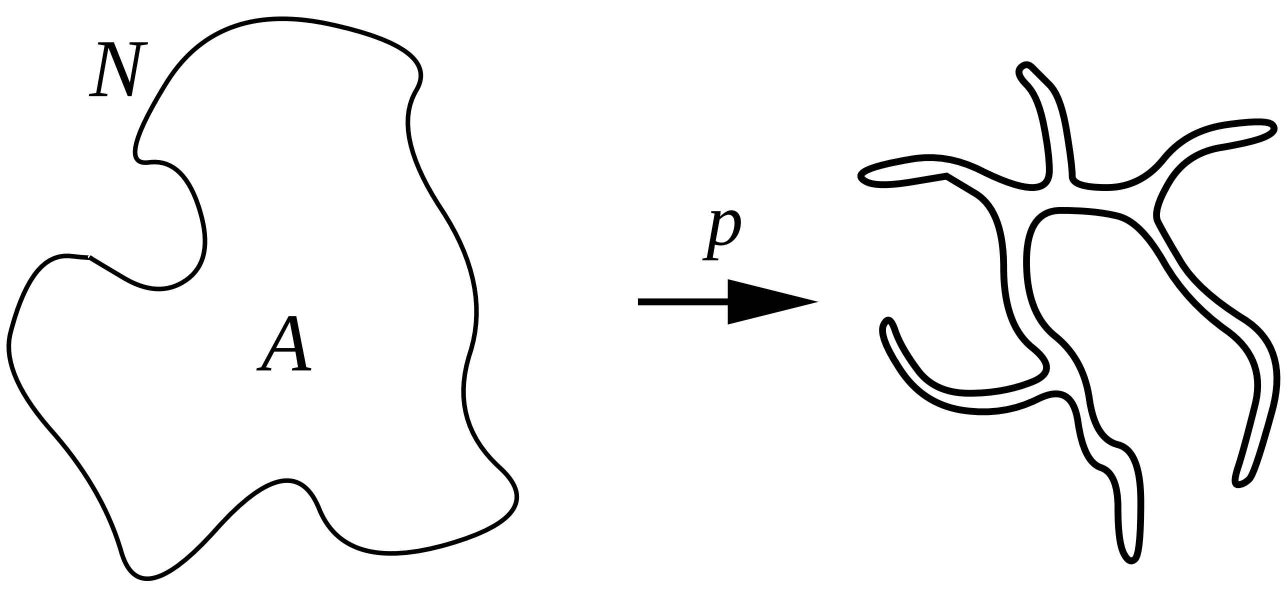
Exact RG flow

- gauge coupling + SUSY $\Rightarrow$

$$dp/d\ell = -\beta(p) = 2p \qquad \text{to all orders}$$

- Crossover phenomenon

Self – avoiding loops $\Longrightarrow$ Branched polymers



"quarks" + gauge field $\Rightarrow$ "hadrons"

• Supersymmetric 'hadrons':

$$\begin{aligned}\Phi &\sim \phi^*\phi\\ \bar\chi &\sim \psi^*\phi\\ \chi &\sim \phi^*\psi\\ \omega &\sim \psi^*\psi\end{aligned}$$

• introduce anticommuting coordinates $(\bar\theta, \theta)$ where

$$\begin{aligned}\int d\bar\theta &= \int d\theta = 0\\ \int d\bar\theta d\theta\, \bar\theta\theta &= -\pi^{-1}\end{aligned}$$

and superfield

$$\Psi(r, \bar\theta, \theta) \equiv \Phi + \bar\chi\theta + \chi\bar\theta + \omega\bar\theta\theta$$

then

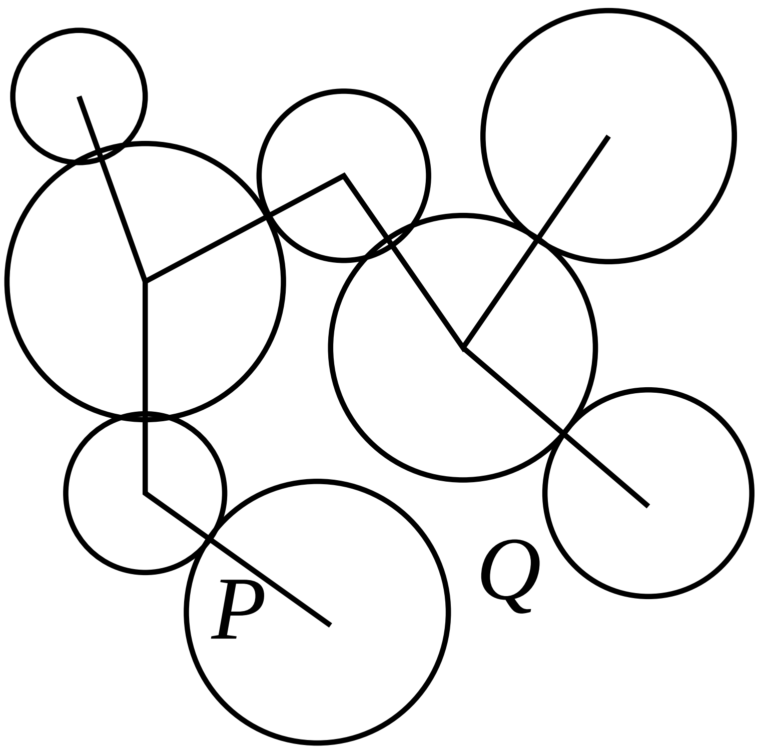
$$S_{\text{eff}} = \int d^2r d\bar\theta d\theta [(\nabla_{SS}\Psi)^2 + V(\Psi)]$$

• exhibits SUSY under transformations which keep $|r|^2 + \bar\theta\theta$ invariant $\Rightarrow$ dimensional reduction (Parisi-Sourlas, 1982).

Exact Dimensional Reduction for Branched Polymers

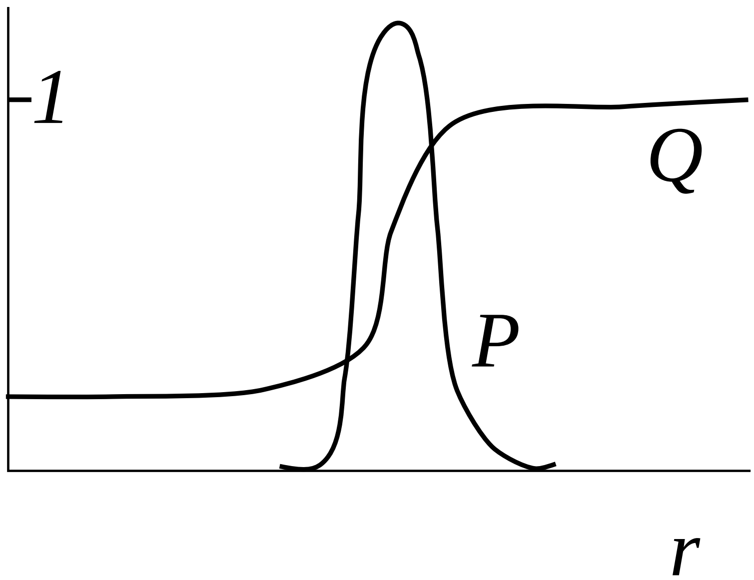
(Brydges and Imbrie 2001)

- ‘sticky ball model’ of branched polymers:



$$Z_{BP}(z) = \sum_T z^{|T|} \int \prod_j d^d r_j \prod_{|j-k|=1} P((r_j - r_k)^2) \prod_{|j-k|>1} Q((r_j - r_k)^2)$$

Form of P and Q:



• **Theorem** (Brydges-Imbrie): if $P(r^2) = Q'(r^2)$ then

$$Z_{BP}(z) = -(1/\pi) \ln Z(-\pi z)$$

where

$$Z(z) = \sum_N \frac{z^N}{N!} \int \prod_j d^{d-2} r_j \prod_{jk} Q((r_j - r_k)^2)$$

is the grand partition function for a repulsive gas in d-**2** dimensions, with

$$Q(r^2) = \exp\left(-\beta V(r^2) \right)$$

Idea of proof: both sides are equivalent to a *supersymmetric* gas

$$Z_{SUSY}(z) = \sum_N \frac{z^N}{N!} \int \prod_j d^d r_j d\theta_j d\bar{\theta}_j \prod_{j<k} Q(r_{jk}^2 + \bar{\theta}_{jk}\theta_{jk})$$

where $r_{jk} = r_j - r_k$, etc.

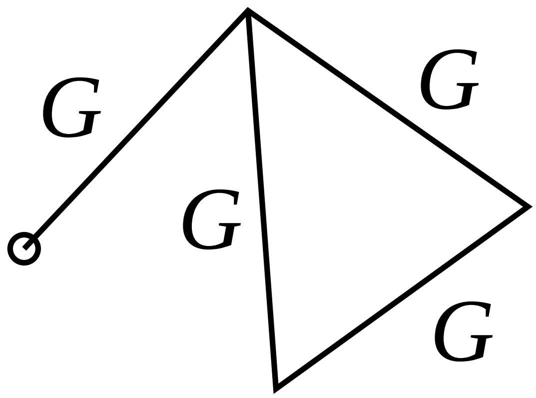
• $Q(r_{jk}^2 + \bar{\theta}_{jk}\bar{\theta}_{jk}) = Q(r_{jk}^2) + \bar{\theta}_{jk}\theta_{jk}P(r_{jk}^2)$

• grassmann integrations $\Rightarrow$ all terms with $N > 0$ vanish, so $Z_{SUSY} = 1$: consider 1-point function (particle density) where one particle is fixed at $\mathbf{0}$. Grassmann integrations eliminate all diagrams with closed loops of P's $\Rightarrow$ sum over *trees*, rooted at $\mathbf{0}$, with weights exactly as in the 'sticky-ball' model.

• Alternatively, let $Q = 1 - G$ and make the Mayer cluster expansion, in powers of

$$G(r_{jk}^2 + \bar{\theta}_{jk}\theta_{jk}) = \int_0^\infty d\alpha_{jk} \tilde{G}(\alpha_{jk})\, e^{-\alpha_{jk}(r_{jk}^2 + \bar{\theta}_{jk}\theta_{jk})}$$

$\Rightarrow$ connected cluster diagrams.



• integrals over the co-ordinates have the form

$$\begin{aligned} &\int \prod_j d^d r_j d\bar{\theta}_j d\theta_j \, e^{-\sum_{jk} r_j M_{jk} r_k} \, e^{-\sum_{jk} \bar{\theta}_j M_{jk} \theta_k} \\ = \; & (-1)^N (\det M)^{-d/2} (\det M) \\ = \; & (-1)^N (\det M)^{-(d-2)/2} \end{aligned}$$

- the cluster expansion for a non-SUSY gas in $d-2$ dimensions, with fugacity $-z$.

• in particular, for $d-2=0$,

$$\begin{aligned} Z(-z) &= \sum_N \frac{(-z)^N}{N!}\, e^{-\frac{1}{2}V(0)N(N-1)} \\ &= \sum_N \frac{(-z')^N}{N!} \int e^{-\frac{1}{2V(0)}t^2+iNt} dt \\ &= \int e^{-(\frac{1}{2}t^2+\tilde{z}\, e^{it})/V(0)} dt \end{aligned}$$

where $\tilde{z} = zV(0)e^{V(0)/2}$.

• this is an entire function of $\tilde{z}$, so the only singularities of $Z_{2dBP}(z) \sim \ln Z(-z)$ are at the zeroes of $Z \Rightarrow \alpha_{BP} = 2$ in $d=2$.

• but there is another interesting limit: consider $V(0) \to 0$: expand about critical point $it = 1$, $\tilde{z} = \tilde{z}_c = e^{-1}$:

$$Z(-z) \sim \int e^{(i(\tilde{z}_c - \tilde{z})t + it^3/3)/V(0)}\, dt$$

– an Airy integral!!

• Exact RG equation:

$$P = Q'(r^2) \sim (1 - e^{-V(0)})\delta(r^2 - a^2)$$

so under rescaling $a \to ae^{\ell}$ in d dimensions, $(1 - e^{-V(0)}) \to (1 - e^{-V(0)})\, e^{2\ell}$.

So

$$\begin{aligned} dp/d\ell &= 2p \\ dV(0)/d\ell &= 2V(0) \quad \text{as } V(0) \to 0 \end{aligned}$$

$\Rightarrow$ conjecture the correspondence

$$\begin{aligned} V(0) &\propto p \qquad \text{as } p \to 0 \\ \tilde{z}_c - \tilde{z} &\propto x_c - x \end{aligned}$$

$$G^{(r)}(x,p) \sim \frac{\int t e^{(i(x_c - x)t + it^3/3)/p}\, dt}{\int e^{(i(x_c - x)t + it/3)/p}\, dt}$$

Rescale $t \to p^{1/3} t$:

$$G^{(r)}(x,p) \sim p^{1/3} F((x_c - x)p^{-2/3})$$

$\Rightarrow \alpha = \frac{1}{2}$, $\nu = \frac{3}{4}$, and $F(u) \propto \mathrm{Ai}'(u)/\mathrm{Ai}(u)$.

• Richard, Guttmann and Jensen confirmed Airy form by numerical estimates of moment ratios

$$\langle A^n \rangle / \langle A \rangle^n$$

where

$$\langle A^n \rangle \sim (\partial/\partial p)^n \, G^{(r)}(x,p)|_{p=0}$$

• Airy function function first conjectured by RGJ on basis of assumed q-algebraic equation satisfied by $G^{(r)}(x,p)$.

Comments and Puzzles

- first example of a scaling function of two thermodynamic variables near a non-trivial isotropic critical point;

- gives a new explanation of why $\nu_{SAW} = \frac{3}{4}$ in $d = 2$: can we make it rigorous?

- can generalise $t^3 \to t^{k+2}$: for $p \to 0$ these lead to new $2d$ multicritical points [physical interpretation as yet obscure, but scaling dimensions agree with twisted $N = 2$ supersymmetric CFTs.]

- a valuable lesson in the unity of theoretical physics!