THE STRESS TENSOR IN QUENCHED RANDOM SYSTEMS

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Outline.

1. The stress tensor in random systems considered as deformed pure systems.
2. Correlators of the stress tensor at a random fixed point: expectations from the replica approach.
3. Partition function on a torus.
4. How the stress tensor enters into correlation functions: subtleties with Kac operators.

• Many results generalise to arbitrary dimension, but take $d = 2$ for simplicity.

• Some of this material in JC, cond-mat/9911024. Some overlap with Gurarie and Ludwig, cond-mat/9911392, but differences in detail.
Quenched random systems as a deformed pure system.

\[ S = S_P + \sum_{i=1}^{N} \int h_i(r)\Phi_i(r)d^2r \]

- \( S_P \) is a non-random CFT; \( \Phi_i(r) \) a set of primary fields (assumed to be scalars – can generalise to vector couplings).

- \( h_i(r) \) quenched random variables, \( \overline{h_i(r)} = 0 \), 
  \( \overline{h_i(r')h_j(r'')} = \lambda_{ij}\delta(r' - r'') \) – take \( N = 1 \) for simplicity.

- interested in RG flow: \( S_P \implies \) (random fixed point).

- the perturbation is not necessarily small: idea is to see how objects in \( S_P \) deform in the full theory. One of the tools will be replica group theory (cf. atomic physics).
Recall **deformed CFT in pure systems** (Zamolodchikov, 1986).

\[
S = S_0 - \lambda \int \Phi(r) d^2r
\]

\[
\delta T(z, \bar{z}) = \lambda \int_{|z' - z| > a} d^2z' T(z) \cdot \Phi(z', \bar{z}')
\]

\[
\partial_{\bar{z}} T(z, \bar{z}) = 
\lambda \int d^2z' \delta(|z - z'|^2 - a^2)(z - z')
\left( \frac{\Delta}{(z - z')^2} \Phi(z, \bar{z}) + \frac{1 - \Delta}{(z - z')} \partial_z \Phi(z, \bar{z}) + \cdots \right)
\]

\[
= -\partial_{\bar{z}} \Theta
\]

where

\[
\Theta = -\pi \lambda (1 - \Delta) \Phi
\]

\[
\propto -\lambda (d - x_\Phi) \Phi \quad \text{(general } d)\]

• this assumes that no additional renormalization imposed (unnecessary if \( x_\Phi < d \)), but in general

\[
\Theta \propto \beta(\lambda) \Phi_R.
\]
Now do this for \textit{random} coupling $\lambda \to h(z, \bar{z})$:

$$\partial_z T = \int d^2z' \delta(|z - z'|^2 - a^2)(z - z')h(z', \bar{z}') \left( \frac{\Delta}{(z - z')^2} \Phi(z, \bar{z}) + \frac{1 - \Delta}{(z - z')} \partial_z \Phi(z, \bar{z}) + \cdots \right)$$

but now $h(z', \bar{z}')$ is white noise. Result:

$$\partial_z T + \partial_z \Theta = K$$

where

$$\Theta(z, \bar{z}) = -\pi \left( \frac{1}{2} - \Delta \Phi \right) h(z, \bar{z}) \Phi(z, \bar{z}) \left[ \propto (d - \frac{d}{2} - x_{\Phi}) h \Phi \right]$$

$$K = \frac{1}{2} \pi (h \partial_z \Phi - \Phi \partial_z h)$$

- $T$ and $\Theta$ are the components of the stress tensor for a \textit{given realization} of randomness, not the quenched average!
- $\frac{d}{2}$ comes from the white noise nature of $h(r)$.
- $\partial h$ interpreted by integrating by parts in correlators
- similar equation relating $\mathcal{T}$ to the \textit{same} $\Theta$: so $T_{zz} = T_{z\bar{z}} \propto \Theta$, even in the random system: local rotational symmetry is preserved [results slightly modified if the coupling is to a random vector].
Replica formulation.

\[
\overline{Z^n} = \int \mathcal{D}h e^{-(1/2\lambda) \int h^2 d^2r} \text{Tr} e^{-\sum_a S_{P_a} + \int h \sum_a \Phi_a d^2r} \\
= \text{Tr} \int \mathcal{D}h e^{-(1/2\lambda) \int (h - \sum_a \Phi_a)^2 d^2r} e^{-\sum_a S_{P_a} + \frac{1}{2\lambda} \int \sum_{a \neq b} \Phi_a \Phi_b d^2r}
\]

which has the form of a translationally invariant perturbed CFT.

Replica theory has a stress tensor \( \mathcal{T} \) which is a deformation of \( \sum_a T_a \), so

\[
\partial_z \mathcal{T} + \partial_{\vartheta} \vartheta = 0
\]

where \( \vartheta = -\frac{1}{2\lambda} \pi (1 - 2\Delta) \sum_{a \neq b} \Phi_a \Phi_b \).

At the new fixed point, \( \vartheta = 0 \), and

\[
\langle \mathcal{T}(z) \mathcal{T}(0) \rangle = c(n)/2z^4
\]

where, by the \( c \)-theorem sum rule,

\[
c(n) - nc_P = -(12/\pi) \int r^2 \langle \vartheta(r) \vartheta(0) \rangle c d^2r
\]
Interpretation:

\[
\langle \mathcal{T} \mathcal{T} \rangle = \sum_{a,b} \langle T_a T_b \rangle = n \langle T_1 T_1 \rangle + n(n - 1) \langle T_1 T_2 \rangle
\]

\[
\sim n \left( \langle \mathcal{T} \mathcal{T} \rangle - \langle T \rangle \langle T \rangle \right)
\]

so that, at the random fixed point

\[
\langle \mathcal{T} \mathcal{T} \rangle_c = c_{\text{eff}}/2z^4
\]

where \( c_{\text{eff}} = c'(0) \), and

\[
\delta c_{\text{eff}} = -3\pi \lambda^2 (1 - 2\Delta)^2 \lim_{n \to 0} (1/n) \sum_{a \neq b} \sum_{c \neq d} \int r^2 \langle \Phi_a(r) \Phi_b(r) \Phi_c(0) \Phi_d(0) \rangle_c d^2 r
\]

\[
= -12\pi (1 - 2\Delta)^2 \int r^2 \overline{h(r)h(0)\Phi(\Phi(0))}_c d^2 r
\]

\[
= -\frac{12\pi (1 - 2\Delta)^2}{\text{area}} \int_{r_1 r_2} h(r_1) h(r_2) \langle \Phi(r_1) \Phi(r_2) \rangle_c d^2 r_1 d^2 r_2
\]

NB no obvious positivity: expect \( h\Phi > 0 \) but above involves \( h(\Phi - \langle \Phi \rangle) \).
**But** there \( n - 1 \) other independent components of the deformed stress tensor:

\[
\mathbf{T} = \sum_a T_a \\
\bar{T}_a = T_a - (1/n) \mathbf{T}
\]

where \( \Sigma_a \bar{T}_a = 0 \).

- Combinations are chosen to transform according to irreps of \( S_n \) - should deform into conformal fields at the new fixed point, with scaling dimensions \((2 + \delta(n), \delta(n))\) (can check in perturbation theory that \( \delta \neq 0 \)).

In the undeformed theory,

\[
\langle \bar{T}_a \bar{T}_b \rangle = \left( \delta_{ab} - \frac{1}{n} \right) \frac{c}{2z^4}
\]

so choose, at the new fixed point,

\[
\langle \bar{T}_a \bar{T}_b \rangle = \left( \delta_{ab} - \frac{1}{n} \right) \frac{c(n)}{2n} \frac{1}{z^4(zz)^{2\delta(n)}}
\]

Then

\[
\frac{\langle T \rangle}{\langle T \rangle} = \lim_{n \to 0} \langle T_1 T_2 \rangle \\
= \lim_{n \to 0} \langle (\bar{T}_1 + (1/n) \mathbf{T})(\bar{T}_2 + (1/n) \mathbf{T}) \rangle \\
= \lim_{n \to 0} \frac{c'(0)}{2z^4} \left( -\frac{1}{n} (zz)^{-2\delta(n)} + \frac{1}{n} \right) \\
= \frac{\tilde{c}_{\text{eff}}}{2z^4} \ln(zz)
\]

where

\[
\tilde{c}_{\text{eff}} = 2c'(0)\delta'(0)
\]
$\overline{T}_a$ is not conserved: in fact

$$\partial_z \overline{T}_a + \partial_z \overline{\vartheta}_a = K_a$$

where

$$\overline{\vartheta}_a = -\pi \lambda \left( \frac{1}{2} - \Delta \right) \Phi_a \sum_{c \neq a} \Phi_c$$

$$K_a = \frac{1}{2} \pi \lambda \sum_{b \neq a} \left( \Phi_a \partial_z \Phi_b - \Phi_b \partial_z \Phi_a \right)$$

- This is equivalent to the previous equation for a fixed $h(r)$ by the substitution $\lambda \sum_b \Phi_b \to h(r)$.

- From this can derive a sum rule for $\delta \tilde{c}_{\text{eff}}$ in terms of suitably averaged correlators of $\Phi$ (but no positivity).

- In a general renormalization scheme

$$\overline{\vartheta}_a = -\frac{1}{2} \pi \left( \beta(\lambda) + \delta(n) \right) \sum_{c \neq a} (\Phi_a \Phi_c)_R - (1/n)\vartheta$$

so that $\overline{\vartheta}_a = O(n)$ at the random fixed point.
The torus partition function.

Torus partition function encodes operator content - $\mathcal{T}$, $\mathcal{T}_a$, etc. For general $n$,

$$Z^n = (q\bar{q})^{-c(n)/24} \left( 1 + q^2 + (n - 1)q^{2+\delta(n)}\bar{q}^\delta(n) + \bar{q}^2 + \right)$$

$$+ (n - 1)q^{\delta(n)}\bar{q}^{2+\delta(n)} + q^2 \bar{q}^2 + \cdots$$

- This should equal 1 when $n = 0$, so there must be enormous cancellations!

- Massive degeneracy of Virasoro primaries as $n \to 0$ - extended algebra (supersymmetry???)

- Quenched free energy $\overline{\ln Z} = (\partial / \partial n)|_{n=0} Z^n$

$$= -c_{\text{eff}} \ln(q\bar{q}) - \delta'(0)(q^2 + \bar{q}^2) \ln(q\bar{q}) + \cdots$$

- Where $\delta'(0) = \tilde{c}_{\text{eff}}/2c_{\text{eff}}$.

Questions:

- How does modular invariance work?

- Boundary states?
Operator product expansions.

The “c \to 0 catastrophe”: for primary $\phi$ in any CFT

$$\phi(z, \bar{z}) \phi(0, 0) = \frac{a_\phi}{z^{2\Delta} \bar{z}^{2\bar{\Delta}}} \left( 1 + \frac{2\Delta}{c} z^2 T + \cdots + \frac{4\Delta \bar{\Delta}}{c^2} z^2 \bar{z}^2 (T \bar{T}) + \cdots \right)$$

so, in the 4-point function

$$\langle \phi \phi \phi \phi \rangle \propto a_\phi^2 (1 + (2\Delta/c)^2 (c/2) \eta^2 + \cdots + O(1/c^4) c^2 (\eta \bar{\eta})^2 + \cdots)$$

- a problem as $c \to 0$. Three possible resolutions:

1. other operators in $\cdots$ cancel the divergence;
2. $a_\phi \to 0$ as $c \to 0$;
3. $(\Delta, \bar{\Delta}) \to (0, 0)$ as $c \to 0$. 
Replica models.
Let \( \Phi = \Sigma_a \Phi_a, \bar{\Phi}_a = \Phi_a - (1/n)\Phi \). In the pure theory, the OPEs are schematically

\[
\bar{\Phi}_a \cdot \bar{\Phi}_a = (1 - 1/n)(z\bar{z})^{-4\Delta} \left( 1 + \frac{2\Delta}{cn} z^2 \mathfrak{T} + \frac{2\Delta}{c} z^2 \bar{\mathfrak{T}}_a + \cdots \right)
\]

\[
\Phi \cdot \Phi = n(z\bar{z})^{-4\Delta} \left( 1 + \frac{2\Delta}{cn} z^2 \mathfrak{T} + \frac{2\Delta^2}{(cn)^2} (z\bar{z})^2 \mathfrak{T} \bar{\mathfrak{T}} + \right.
\]

\[
\left. + \frac{2\Delta^2}{c^2} (z\bar{z})^2 \sum_a \bar{\mathfrak{T}}_a \bar{\mathfrak{T}}_a + \cdots \right)
\]

which deform into

\[
\bar{\Phi}_a \cdot \bar{\Phi}_a = (1 - 1/n)(z\bar{z})^{-4\Delta} \Phi \left( 1 + \frac{2\Delta \bar{\Phi}}{c(n)} z^2 \mathfrak{T} + \right.
\]

\[
\left. + \text{const} \bar{z}^2 (z\bar{z})^\delta(n) \bar{\mathfrak{T}}_a + \cdots \right)
\]

\[
\Phi \cdot \Phi = n(z\bar{z})^{-4\Delta} \Phi \left( 1 + \frac{2\Delta \Phi}{c(n)} z^2 \mathfrak{T} + \frac{2\Delta^2 \Phi}{c(n)^2} (z\bar{z})^2 \mathfrak{T} \bar{\mathfrak{T}} + \right.
\]

\[
\left. + \text{const} (z\bar{z})^{2+\delta_2(n)} \mathcal{M} + \cdots \right)
\]

where \( \mathcal{M} \) is a new primary operator with dimensions \((2 + \delta_2(n), 2 + \delta_2(n))\). Thus \( \bar{\Phi} \) and \( \Phi \) resolve the “\( c \to 0 \) catastrophe” differently. 4-point functions have form

\[
\langle \bar{\Phi}_a \bar{\Phi}_a \bar{\Phi}_a \bar{\Phi}_a \rangle \sim 1 + \delta'(0) \eta^2 \ln(\eta\bar{\eta}) + \cdots
\]

\[
\langle \Phi \Phi \Phi \Phi \rangle \sim c \left( 1 + \eta^2 + \cdots + \delta'_2(0)(\eta\bar{\eta})^2 \ln(\eta\bar{\eta}) + \cdots \right)
\]
NB $\Phi_a \equiv \tilde{\Phi}_a + (1/n)\Phi$ and $\Phi$ are an example of a logarithmic pair: at $c = 0$

$$
\langle \Phi_a(z, \bar{z})\Phi_a(0, 0) \rangle \sim (z\bar{z})^{-4\Delta} \ln(z\bar{z})
$$

$$
\langle \Phi_a(z, \bar{z})\Phi(0, 0) \rangle \sim (z\bar{z})^{-4\Delta}
$$

$$
\langle \Phi(z, \bar{z})\Phi(0, 0) \rangle = 0
$$

- **Kac operators** are examples of the second solution.

- Def.: a Kac operator $\phi$ has scaling dimensions at some fixed position in the Kac table for a range of $c$ including 0.

- Only other Kac operators can appear in the OPE $\phi \cdot \phi$: this excludes a companion of $\mathcal{T}$, which would have dimension $(2 + \delta, \delta)$. Hence $a_\phi \to 0$ as $c \to 0$. But $\mathcal{M}$ with dimensions $(2 + \delta_2, 2 + \delta_2)$ does exist!

- if we choose $a_\phi \propto c^p$, the $2N$-point connected correlator goes like $c^{N(p-1)+1}$, so it is natural to take $p = 1$.

- this is exactly what happens in physical examples of percolation ($(Q \to 1)$-Potts model) or self-avoiding walks ($O(n \to 0)$ model), where physical quantities are derivatives wrt $c$ of correlators of Kac operators.
Summary

• Stress tensor in a general quenched random system, with a given distribution of impurities, satisfies

\[ \partial_z T + \partial_z \Theta = K \]

with explicit expressions for \( \Theta \) and \( K \).

• at a random fixed point,

\[
\begin{align*}
\langle TT \rangle_c &= \frac{c_{\text{eff}}}{2z^4} \\
\langle TT \rangle &= \left( \frac{\tilde{c}_{\text{eff}}}{2z^4} \right) \ln(z \bar{z})
\end{align*}
\]

• sum rules for the change in \( c_{\text{eff}} \) and \( \tilde{c}_{\text{eff}} \) along a RG trajectory between 2 fixed points, in terms of physically measurable correlators.

• massive degeneracy of operators at \( c = 0 \) – extended symmetry (????) – but the candidates \( \bar{\mathcal{F}} \) for generators are not holomorphic fields!

• some operators solve the “\( c \to 0 \) catastrophe” by having connected correlators which are all \( O(c) \) – this is true of all Kac operators – but the \textit{physics} is in the \( O(c) \) term and is therefore invisible in the theory at \( c = 0 \).