THE STRESS TENSOR IN QUENCHED RANDOM SYSTEMS

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Outline.

1. The stress tensor in random systems considered as deformed pure systems.
2. Correlators of the stress tensor at a random fixed point: expectations from the replica approach.
3. Partition function on a torus.
4. How the stress tensor enters into correlation functions: subtleties with Kac operators.

- Many results generalise to arbitrary dimension, but take $d = 2$ for simplicity.
- Some of this material in JC, cond-mat/9911024. Some overlap with Gurarie and Ludwig, cond-mat/9911392, but differences in detail.

Quenched random systems as a deformed pure system.

$$S = S_P + \sum_{i=1}^{N} \int h_i(r)\Phi_i(r)d^2r$$

- S_P is a non-random CFT; $\Phi_i(r)$ a set of primary fields (assumed to be scalars – can generalise to vector couplings).
- $h_i(r)$ quenched random variables, $\overline{h_i(r)} = 0$, $\overline{h_i(r')h_j(r'')} = \lambda_{ij}\delta(r' - r'')$ – take $N = 1$ for simplicity.
- interested in RG flow: $S_P \Longrightarrow$(random fixed point).
- the perturbation is not necessarily small: idea is to see how objects in S_P deform in the full theory. One of the tools will be replica group theory (cf. atomic physics).

Recall **deformed CFT in pure systems** (Zamolodchikov, 1986).

$$S = S_0 - \lambda \int \Phi(r) d^2 r$$

$$\delta T(z, \bar{z}) = \lambda \int_{|z'-z|>a} d^2 z' T(z) \cdot \Phi(z', \bar{z}')$$

$$\begin{aligned}
\partial_{\bar{z}} T(z, \bar{z}) & \\
&= \lambda \int d^2 z' \delta(|z - z'|^2 - a^2)(z - z') \\
&\quad \left(\frac{\Delta}{(z - z')^2} \Phi(z, \bar{z}) + \frac{1 - \Delta}{(z - z')} \partial_z \Phi(z, \bar{z}) + \cdots \right) \\
&= -\partial_z \Theta
\end{aligned}$$

where

$$\begin{aligned}
\Theta &= -\pi\lambda(1 - \Delta)\Phi \\
&\propto -\lambda(d - x_\Phi)\Phi \qquad \text{(general } d\text{)}
\end{aligned}$$

- this assumes that no additional renormalization imposed (unnecessary if $x_\Phi < d$), but in general $\Theta \propto \beta(\lambda)\Phi_R$.

Now do this for *random* coupling $\lambda \to h(z,\bar{z})$:

$$\partial_{\bar{z}} T = \int d^2z' \delta(|z-z'|^2 - a^2)(z-z')h(z',\bar{z}')$$
$$\left(\frac{\Delta}{(z-z')^2}\Phi(z,\bar{z}) + \frac{1-\Delta}{(z-z')}\partial_z\Phi(z,\bar{z}) + \cdots\right)$$

but now $h(z',\bar{z}')$ is white noise. Result:

$$\partial_{\bar{z}} T + \partial_z \Theta = K$$

where

$$\begin{aligned} \Theta(z,\bar{z}) &= -\pi(\tfrac{1}{2} - \Delta_\Phi)h(z,\bar{z})\Phi(z,\bar{z}) \\ & \quad [\propto (d - \tfrac{d}{2} - x_\Phi)h\Phi] \\ K &= \tfrac{1}{2}\pi(h\partial_z\Phi - \Phi\partial_z h) \end{aligned}$$

- T and Θ are the components of the stress tensor for a *given realization* of randomness, not the quenched average!
- $\frac{d}{2}$ comes from the white noise nature of $h(r)$.
- ∂h interpreted by integrating by parts in correlators
- similar equation relating $\overline{T}$ to the *same* Θ: so $T_{z\bar{z}} = T_{\bar{z}z} \propto \Theta$, even in the random system: local rotational symmetry is preserved [results slightly modified if the coupling is to a random vector].

Replica formulation.

$$
\begin{aligned}
\overline{Z^n} &= \int \mathcal{D}h e^{-(1/2\lambda)\int h^2 d^2r} \mathrm{Tr}\, e^{-\sum_a S_{P,a} + \int h \sum_a \Phi_a d^2r} \\
&= \mathrm{Tr} \int \mathcal{D}h e^{-(1/2\lambda)\int (h-\sum_a \Phi_a)^2 d^2r} e^{-\sum_a S_{P,a} + \frac{1}{2}\lambda \int \sum_{a\neq b} \Phi_a \Phi_b d^2r}
\end{aligned}
$$

which has the form of a translationally invariant perturbed CFT.

Replica theory has a stress tensor $\mathcal{T}$ which is a deformation of $\sum_a T_a$, so

$$\partial_{\bar{z}} \mathcal{T} + \partial_z \vartheta = 0$$

where $\vartheta = -\frac{1}{2}\lambda\pi(1-2\Delta)\sum_{a\neq b} \Phi_a \Phi_b$.

At the new fixed point, $\vartheta = 0$, and

$$\langle \mathcal{T}(z)\mathcal{T}(0)\rangle = c(n)/2z^4$$

where, by the c-theorem sum rule,

$$c(n) - nc_P = -(12/\pi)\int r^2 \langle \vartheta(r)\vartheta(0)\rangle_c d^2r$$

Interpretation:

$$\begin{aligned}\langle\mathcal{T}\mathcal{T}\rangle &= \sum_{a,b}\langle T_aT_b\rangle = n\langle T_1T_1\rangle + n(n-1)\langle T_1T_2\rangle \\ &\sim n\left(\overline{\langle TT\rangle} - \overline{\langle T\rangle\langle T\rangle}\right)\end{aligned}$$

so that, at the random fixed point

$$\overline{\langle TT\rangle_c} = c_{\rm eff}/2z^4$$

where $c_{\rm eff} = c'(0)$, and

$$\begin{aligned}\delta c_{\rm eff} &= -3\pi\lambda^2(1-2\Delta)^2\lim_{n\to 0}(1/n) \\ &\quad \sum_{a\neq b}\sum_{c\neq d}\int r^2\langle\Phi_a(r)\Phi_b(r)\Phi_c(0)\Phi_d(0)\rangle_c d^2r \\ &= -12\pi(1-2\Delta)^2\int r^2\overline{h(r)h(0)\langle\Phi(r)\Phi(0)\rangle_c}d^2r \\ &= -\frac{12\pi(1-2\Delta)^2}{\text{area}}\int r_{12}^2h(r_1)h(r_2)\langle\Phi(r_1)\Phi(r_2)\rangle_c d^2r_1d^2r_2\end{aligned}$$

NB no obvious positivity: expect $h\Phi > 0$ but above involves $h(\Phi - \langle\Phi\rangle)$.

But there $n-1$ other independent components of the deformed stress tensor:

$$\begin{aligned}\mathcal{T} &= \sum_a T_a \\ \widetilde{\mathcal{T}}_a &= T_a - (1/n)\mathcal{T}\end{aligned}$$

where $\Sigma_a \widetilde{\mathcal{T}}_a = 0$.

- Combinations are chosen to transform according to irreps of S_n - should deform into conformal fields at the new fixed point, with scaling dimensions $(2+\delta(n), \delta(n))$ (can check in perturbation theory that $\delta \neq 0$.)

In the undeformed theory,

$$\langle \widetilde{\mathcal{T}}_a \widetilde{\mathcal{T}}_b \rangle = \left(\delta_{ab} - \frac{1}{n}\right) \frac{c}{2z^4}$$

so choose, at the new fixed point,

$$\langle \widetilde{\mathcal{T}}_a \widetilde{\mathcal{T}}_b \rangle = \left(\delta_{ab} - \frac{1}{n}\right) \frac{c(n)}{2n} \frac{1}{z^4 (z\bar{z})^{2\delta(n)}}$$

Then

$$\begin{aligned}\overline{\langle T \rangle \langle T \rangle} &= \lim_{n\to 0} \langle T_1 T_2 \rangle \\ &= \lim_{n\to 0} \langle (\widetilde{\mathcal{T}}_1 + (1/n)\mathcal{T})(\widetilde{\mathcal{T}}_2 + (1/n)\mathcal{T}) \rangle \\ &= \lim_{n\to 0} \frac{c'(0)}{2z^4} \left(-\frac{1}{n}(z\bar{z})^{-2\delta(n)} + \frac{1}{n}\right) \\ &= \frac{\tilde{c}_{\text{eff}}}{2z^4} \ln(z\bar{z})\end{aligned}$$

where

$$\tilde{c}_{\text{eff}} = 2c'(0)\delta'(0)$$

$\widetilde{\mathcal{T}}_a$ is not conserved: in fact

$$\partial_{\bar z}\widetilde{\mathcal{T}}_a + \partial_z \widetilde{\vartheta}_a = K_a$$

where

$$\begin{aligned} \widetilde{\vartheta}_a &= -\pi\lambda(\tfrac{1}{2} - \Delta)\Phi_a \sum_{c\neq a} \Phi_c \\ K_a &= \tfrac{1}{2}\pi\lambda \sum_{b\neq a} (\Phi_a \partial_z \Phi_b - \Phi_b \partial_z \Phi_a) \end{aligned}$$

- This is equivalent to the previous equation for a fixed $h(r)$ by the substitution $\lambda\,\Sigma_b\,\Phi_b \to h(r)$.
- from this can derive a sum rule for $\delta\tilde{c}_{\rm eff}$ in terms of suitably averaged correlators of Φ (but no positivity).
- in a general renormalization scheme

$$\widetilde{\vartheta}_a = -\frac{1}{2}\pi(\beta(\lambda) + \delta(n)) \sum_{c\neq a} (\Phi_a\Phi_c)_R - (1/n)\vartheta$$

  so that $\widetilde{\vartheta}_a = O(n)$ at the random fixed point.

The torus partition function.

Torus partition function encodes operator content - $\mathcal{T}$, $\widetilde{\mathcal{T}}_a$, etc. For general n,

$$\begin{aligned}\overline{Z^n} \; &= \; (q\bar{q})^{-c(n)/24} \\ &\Big(1 + q^2 + (n-1)q^{2+\delta(n)}\bar{q}^{\delta(n)} + \bar{q}^2 + \\ &(n-1)q^{\delta(n)}\bar{q}^{2+\delta(n)} + q^2\bar{q}^2 + \cdots\Big)\end{aligned}$$

- This should equal 1 when $n = 0$, so there must be enormous cancellations!
- Massive degeneracy of Virasoro primaries as $n \to 0$ - extended algebra (supersymmetry???)
- quenched free energy $\overline{\ln Z} = (\partial/\partial n)|_{n=0}\overline{Z^n}$

$$= -c_{\text{eff}} \ln(q\bar{q}) - \delta'(0)(q^2 + \bar{q}^2) \ln(q\bar{q}) + \cdots$$

- where $\delta'(0) = \tilde{c}_{\text{eff}}/2c_{\text{eff}}$.

Questions:

- how does modular invariance work?
- boundary states?

Operator product expansions.

The "$c \to 0$ catastrophe": for primary ϕ in any CFT

$$\phi(z,\bar{z}) \quad \phi(0,0) = \frac{a_\phi}{z^{2\Delta}\bar{z}^{2\bar{\Delta}}}\left(1 + \frac{2\Delta}{c} z^2 T + \cdots + \frac{4\Delta\bar{\Delta}}{c^2} z^2 \bar{z}^2 (T\overline{T}) + \cdots\right)$$

so, in the 4-point function

$$\langle\phi\phi\phi\phi\rangle \propto a_\phi^2 (1 + (2\Delta/c)^2 (c/2)\eta^2 + \cdots + O(1/c^4) c^2 (\eta\bar{\eta})^2 + \cdots)$$

- a problem as $c \to 0$. Three possible resolutions:

1. other operators in $\cdots$ cancel the divergence;
2. $a_\phi \to 0$ as $c \to 0$;
3. $(\Delta, \bar{\Delta}) \to (0, 0)$ as $c \to 0$.

Replica models.

Let $\Phi = \Sigma_a \Phi_a$, $\widetilde{\Phi}_a = \Phi_a - (1/n)\Phi$. In the pure theory, the OPEs are schematically

$$\begin{aligned}
\widetilde{\Phi}_a \cdot \widetilde{\Phi}_a &= (1-1/n)(z\bar{z})^{-4\Delta}\left(1 + \frac{2\Delta}{cn} z^2 \mathcal{T} + \frac{2\Delta}{c} z^2 \widetilde{\mathcal{T}}_a + \cdots\right) \\
\Phi \cdot \Phi &= n(z\bar{z})^{-4\Delta}\left(1 + \frac{2\Delta}{cn} z^2 \mathcal{T} + \frac{2\Delta^2}{(cn)^2}(z\bar{z})^2 \mathcal{T}\overline{\mathcal{T}} + \right. \\
&\qquad \left. + \frac{2\Delta^2}{c^2}(z\bar{z})^2 \sum_a \widetilde{\mathcal{T}}_a \overline{\widetilde{\mathcal{T}}}_a + \cdots\right)
\end{aligned}$$

which deform into

$$\begin{aligned}
\widetilde{\Phi}_a \cdot \widetilde{\Phi}_a &= (1-1/n)(z\bar{z})^{-4\Delta_{\widetilde{\Phi}}}\left(1 + \frac{2\Delta_{\widetilde{\Phi}}}{c(n)} z^2 \mathcal{T} + \right. \\
&\qquad \left. + \mathrm{const}\, z^2 (z\bar{z})^{\delta}(n) \widetilde{\mathcal{T}}_a + \cdots\right) \\
\Phi \cdot \Phi &= n(z\bar{z})^{-4\Delta_\Phi}\left(1 + \frac{2\Delta_\Phi}{c(n)} z^2 \mathcal{T} + \frac{2\Delta_\Phi^2}{c(n)^2}(z\bar{z})^2 \mathcal{T}\overline{\mathcal{T}} + \right. \\
&\qquad \left. + \mathrm{const}(z\bar{z})^{2+\delta_2(n)} \mathcal{M} + \cdots\right)
\end{aligned}$$

where $\mathcal{M}$ is a new primary operator with dimensions $(2+\delta_2(n), 2+\delta_2(n))$. Thus $\widetilde{\Phi}$ and Φ resolve the "$c \to 0$ catastrophe" differently. 4-point functions have form

$$\begin{aligned}
\langle \widetilde{\Phi}_a \widetilde{\Phi}_a \widetilde{\Phi}_a \widetilde{\Phi}_a \rangle &\sim 1 + \delta'(0)\eta^2 \ln(\eta\bar{\eta}) + \cdots \\
\langle \Phi\Phi\Phi\Phi \rangle &\sim c\left(1 + \eta^2 + \cdots + \delta_2'(0)(\eta\bar{\eta})^2 \ln(\eta\bar{\eta}) + \cdots\right)
\end{aligned}$$

NB $\Phi_a \equiv \widetilde{\Phi}_a + (1/n)\Phi$ and Φ are an example of a *logarithmic pair*: at $c = 0$

$$\begin{aligned} \langle \Phi_a(z,\bar{z})\Phi_a(0,0)\rangle &\sim (z\bar{z})^{-4\Delta}\ln(z\bar{z}) \\ \langle \Phi_a(z,\bar{z})\Phi(0,0)\rangle &\sim (z\bar{z})^{-4\Delta} \\ \langle \Phi(z,\bar{z})\Phi(0,0)\rangle &= 0 \end{aligned}$$

- **Kac operators** are examples of the second solution.
- Def.: a Kac operator ϕ has scaling dimensions at some fixed position in the Kac table for a range of c including 0.
- Only other Kac operators can appear in the OPE $\phi \cdot \phi$: this excludes a companion of $\mathcal{T}$, which would have dimension $(2+\delta, \delta)$. Hence $a_\phi \to 0$ as $c \to 0$. But $\mathcal{M}$ with dimensions $(2+\delta_2, 2+\delta_2)$ does exist!
- if we choose $a_\phi \propto c^p$, the $2N$-point connected correlator goes like $c^{N(p-1)+1}$, so it is natural to take $p = 1$.
- this is exactly what happens in physical examples of percolation (($Q \to 1$)-Potts model) or self-avoiding walks (O($n \to 0$) model), where physical quantities are *derivatives* wrt c of correlators of Kac operators.

Summary

- Stress tensor in a general quenched random system, with a given distribution of impurities, satisfies

$$\partial_{\bar{z}} T + \partial_z \Theta = K$$

  with explicit expressions for Θ and K.

- at a random fixed point,

$$\begin{aligned} \overline{\langle TT \rangle_c} &= c_{\text{eff}}/2z^4 \\ \overline{\langle TT \rangle} &= (\tilde{c}_{\text{eff}}/2z^4) \ln(z\bar{z}) \end{aligned}$$

- sum rules for the change in c_{eff} and $\tilde{c}_{\text{eff}}$ along a RG trajectory betwen 2 fixed points, in terms of physically measurable correlators.

- massive degeneracy of operators at $c = 0$ – extended symmmetry (???) – but the candidates $\widetilde{\mathcal{T}}$ for generators are not holomorphic fields!

- some operators solve the "$c \to 0$ catastrophe" by having connected correlators which are all $O(c)$ – this is true of all Kac operators – but the *physics* is in the $O(c)$ term and is therefore invisible in the theory at $c = 0$.