Conformal Multifractality

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- M. E. Cates and T. Witten, *Phys. Rev. Lett.* **56**, 2497, 1986; *Phys. Rev. B* **35**, 1809, 1987.
- G. F. Lawler and W. Werner, to be published.
- B. Duplantier, *Phys. Rev. Lett.* **81**, 5489, 1998; cond-mat/9908314 and references therein.
- JC, *J. Phys. A* **32**, L177, 1999; cond-mat/9812416

Harmonic measure of a fractal curve γ.

- $\nabla^2\phi = 0$ outside γ, $\phi = 0$ on γ, $\partial\phi/\partial r \sim (1/2\pi r)$ as $r \to \infty$. What is distribution of the flux $\Phi(x)$ onto a point x of γ? [Discretised version: $\Phi(x) = a^{-1}\sum_{|r-x|=a}\phi(r)$.]

- Brownian particle starts at ∞. Pr(particle first hits γ at x) $= \Phi(x)$.

$$\overline{\sum_x \Phi(x)^n / \sum_x 1} \sim (a/R)^{\lambda_n} \qquad (R = \text{ linear dimension of } \gamma.)$$

$$\lambda_0 = 0 \qquad \lambda_1 = D = \text{ fractal dimension}$$

Multiscaling: $\lambda_n \neq n \cdot \lambda_1$.

More general setting: CFT with a fractal boundary

Harmonic problem : $\phi(r_1) = \int' [\mathcal{D}\phi]\phi(r_1)e^{-\frac{1}{2}\int(\nabla\phi)^2 d^2r}$

More generally, we have a CFT outside γ with a boundary condition on γ, with ϕ some conformal field, and ask for

$$\overline{\langle\phi(r)\phi(R')\rangle^n} \sim a^{-nx}R'^{-nx}(a/R')^{\lambda_n}$$

where $|r-\gamma| = a$ and $R' \sim R$.

NB near a *flat* boundary in general we get $a^{-x}R^{-x}(a/R)^{\tilde{x}}$ and x and $\tilde{x}$ are the bulk and boundary scaling dimensions of ϕ.

In a *wedge* of opening angle θ, $\tilde{x} \to \pi\tilde{x}/\theta$. Fractal: distribution of θs?

Basic assumption: conformal invariance of measure in $\overline{\ \cdot\ }$

– valid for the frontier of a Brownian walk, assumed true in continuum limit for self-avoiding walks, boundary of percolation clusters, etc.

Basic idea of calculation: conformally map region exterior to γ into interior of rectangle $\pi \times \ell_{\text{eff}}(\gamma, \ell \equiv \ln(R/a))$. (see Figure).

If γ were flat,

$$\langle \phi(r')\phi(R') \rangle_{\text{rectangle}} \sim e^{-\tilde{x}\ell}$$

In general,

$$\langle \phi(r')\phi(R') \rangle_{\text{rectangle}} \sim e^{-\tilde{x}\ell_{\text{eff}}(\gamma,\ell)}$$

Take nth power and average over γ:

$$e^{-\lambda_n \ell} \sim \overline{e^{-n\tilde{x}\ell_{\text{eff}}(\gamma,\ell)}}$$

How to calculate averages of the form $\overline{e^{-p\ell_{\text{eff}}}}$?

Remark: this is independent of choice of ϕ. Choose $\phi = \phi_M$, the M-leg operator for M mutually avoiding SAWs (Fig. 2). Weighted sum over all such walks from r_1 to r_2

$$= \langle \phi_M(r_1)\phi_M(r_2)\rangle_{\text{bulk}} \sim |r_1 - r_2|^{-2x_M}$$

(with a corresponding boundary exponent $\tilde{x}_M$.)

- sum over M SAWs, from r near γ to R, followed by sum over all γ of linear size $R \sim$ sum over $M+2$ walks from r to R. So

$$e^{-(x_{M+2}-x_2)\ell} \sim \overline{e^{-\tilde{x}_M \ell_{\text{eff}}(\gamma,\ell)}}$$

Recall $e^{-\lambda_n \ell} \sim \overline{e^{-n\tilde{x}\ell_{\text{eff}}(\gamma,\ell)}}$

- Choose M such that $\tilde{x}_M = n\tilde{x}$. Then $\lambda_n = x_{M+2} - x_2$.

What are x_M and $\tilde{x}_M$?

$$\langle \phi_M(0)\phi_M(\ell)\rangle_{\text{rectangle}} \sim e^{-\tilde{x}_M \ell}; \qquad \langle \phi_M(0)\phi_M(\ell)\rangle_{\text{cylinder}} \sim e^{-x_M \ell}$$

Obtain from eigenvalues of transfer matrix of lattice problem $Na \times N\ell a$ using Bethe ansatz (Yung & Batchelor, 1995). Agrees with less rigorous Coulomb gas methods.

$$x_M = \tfrac{3}{16}M^2 - \tfrac{1}{12} \qquad \tilde{x}_M = \tfrac{1}{8}M(3M+2)$$

Final result:

$$\boxed{\lambda_n = \tfrac{1}{48}(\sqrt{1+24n\tilde{x}}+11)(\sqrt{1+24n\tilde{x}}-1)}$$

$f(\alpha)$ interpretation

Distribution of boundary exponents:

$$e^{-\lambda_n \ell} \sim \int e^{f(\alpha)\ell}\, e^{-n\alpha\ell}\, d\alpha$$

where $\ell = \ln(R/a) \gg 1$. Steepest descents: Legendre transform

$$f'(\alpha) = n; \qquad \lambda_n = n \cdot \alpha - f(\alpha); \qquad \partial\lambda_n/\partial n = \alpha$$

$$f(\alpha) = -\frac{1}{12}\frac{(\alpha-3)^2}{(2\alpha-1)}; \qquad (\tilde{x} = 1)$$

Singularity at $\alpha = \frac{1}{2}$: maximum opening angle $\theta = 2\pi$; $f(\alpha) \sim -\alpha/24$ as $\alpha \to \infty$: moments diverge for $n \leq -\frac{1}{24}$.

Distribution of opening angles.

Recall exponent for a wedge of opening angle θ is $\pi\tilde{x}/\theta$. So

$$P(\theta) \sim e^{f(\pi\tilde{x}/\theta)\ell} \sim \exp\left(-\frac{\ell}{24}\left(\sqrt{(\pi/\theta)-(1/2)}-\frac{5}{2\sqrt{(\pi/\theta)-(1/2)}}\right)^2\right)$$

Remark: as $\ell \to \infty$, $P(\theta)$ strongly peaked around *typical* value $\theta = \pi/3$. In the limit all angles are 60° (if measured this way)!

Generalisations and remarks.

- Same result for $\gamma =$ frontier of a Brownian walk, (accessible) hull of a percolation cluster (bridging fjords), ... These all have $D = \frac{4}{3}$.

- Case where γ is generated by a CFT with central charge $c \neq 0$, e.g. boundary of Ising model clusters. Same argument but use as a test operator $\phi_M = M$-leg operator for M Ising cluster boundaries. x_M, $\tilde{x}_M$ depend on c. Result
$$P(\theta, c) = P(\theta, 0) e^{-cg(\theta)\ell} e^{-cg(2\pi-\theta)\ell}$$
where $cg(\theta)\ell$ is contribution to $\ln Z$ from a wedge of angle θ.

- Similar result for coupling to *quenched* random bulk metric:
$$\lambda_n^{\text{bulk}} = \frac{1}{2}\left(\sqrt{1 + 12nx} - 1\right)$$

Conclusions

Exact results for multiscaling exponents at ‘conformally’ invariant fractal boundaries in $d = 2$. Analytic results confirm general form of the multifractal $f(\alpha)$ hypothesis. Results (almost) rigorous for Brownian walks, less so for other problems. Many open challenges!