Measuring Quantum Entanglement

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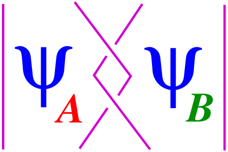

Quantum Entanglement

- Quantum Entanglement is one of the most fascinating and counter-intuitive aspects of Quantum Mechanics
- its existence was first recognised in early work of the pioneers of quantum mechanics[1]
- it is the basis of the celebrated Einstein-Podolsky-Rosen paper[2] which argued that its predictions are incompatible with locality

[1] Schrödinger E (1935). "Discussion of probability relations between separated systems". Mathematical Proceedings of the Cambridge Philosophical Society **31** (4): 555-563. Communicated by M Born

[2] Einstein A, Podolsky B, Rosen N (1935). "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?". Phys. Rev. **47** (10): 777-780.

- this was part of Einstein's programme to refute the probabilistic interpretation of quantum mechanics, due to Born and others

"Die Quantenmechanik ist sehr achtunggebietend. Aber eine innere Stimme sagt mir, daß das noch nicht der wahre Jakob ist. Die Theorie liefert viel, aber dem Geheimnis des Alten bringt sie uns kaum näher. Jedenfalls bin ich überzeugt, daß der Alte nicht würfelt." [3]

"Quantum mechanics is certainly impressive. But an inner voice tells me that it is not yet the real thing. The theory tells us a lot, but it does not bring us any closer to the secrets of the ancients. I, at any rate, am convinced that the old fellow does not play dice."

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- Bell[4] showed that EPR's explanation, involving hidden variables, is inconsistent with the predictions of quantum mechanics – this was subsequently tested experimentally

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Outline

- in this talk I am going to assume that conventional quantum mechanics (and the Copenhagen interpretation) holds and will address the questions:
  - what is quantum entanglement and is there a universal measure of the amount of entanglement?
  - how does this behave for systems with many degrees of freedom?
  - how might it be measured experimentally?

A simple example

- a system consisting of two qubits (spin-$\frac{1}{2}$ particles) with a basis of states

$$(|\uparrow\rangle_A, |\downarrow\rangle_A) \times (|\uparrow\rangle_B, |\downarrow\rangle_B)$$

- Alice observes qubit A, Bob observes qubit B
- the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B\right)$$

  is entangled: before Alice measures σ_A^z, Bob can obtain either result $\sigma_B^z = \pm 1$, but after she makes the measurement the state in B collapses and Bob can only get one result
- moreover this still holds if the subsystems A and B are far apart (the EPR paradox)

- an entangled state is different from a classically correlated state, eg with density matrix

$$\rho = \frac{1}{2}|\uparrow\rangle_A|\uparrow\rangle_B\,{}_A\langle\uparrow|_B\langle\uparrow| + \frac{1}{2}|\downarrow\rangle_A|\downarrow\rangle_B\,{}_A\langle\downarrow|_B\langle\downarrow|$$

- in both cases

$$\langle {\sigma_A}^z {\sigma_B}^z \rangle = 1$$

but for the entangled state

$$\langle {\sigma_A}^x {\sigma_B}^x \rangle = 1$$

while it vanishes for the classically correlated state

Entanglement of pure states

- is there a good way of characterising the degree of entanglement (of pure states)?
- Schmidt decomposition theorem: any state in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written

$$|\Psi\rangle = \sum_j c_j |\psi_j\rangle_A |\psi_j\rangle_B \qquad \text{(S)}$$

  where the states are orthonormal, $c_j > 0$, and $\sum_j c_j^2 = 1$
- the c_j^2 are the eigenvalues of the reduced density matrix

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$

- if there is only one term in (S), $|\Psi\rangle$ is unentangled
- if all the c_j are equal, $|\Psi\rangle$ is maximally entangled

- a suitable measure is the entanglement entropy

$$S_A = -\sum_j c_j^2 \log c_j^2 = -\mathrm{Tr}_A\, \rho_A \log \rho_A = S_B$$

  - it is zero for unentangled states and maximal when all the c_j are equal
  - it is convex: $S_{A_1 \cup A_2} \leq S_{A_1} + S_{A_2}$
  - it is basis-independent

- it increases under Local Operations and Classical Communication

- even for many-body systems it is often computable (analytically, or numerically by density matrix renormalization group methods or matrix product states)

- but it is not the only such measure: eg the Rényi entropies

$$S_A^{(n)} \propto -\log \mathrm{Tr}_A \, {\rho_A}^n$$

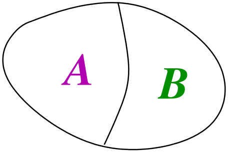
are equally useful, and for different n give information about the whole entanglement spectrum of ρ_A

Entanglement Entropy in Extended Systems

- consider a system whose degrees of freedom are extended in space, e.g. a quantum magnet described by the Heisenberg model with hamiltonian

$$H = \sum_{r,r'} J(r - r')\ \vec{\sigma}(r) \cdot \vec{\sigma}(r')$$

- the temperature is low enough so the system is in the ground state $|0\rangle$ of H



- suppose A is a large but finite region of space: what is the degree of entanglement of the spins within A with the reminder in B?
- since $S_A = S_B$ it can't be $\propto$ the volume of A or B
- in fact in almost all cases we have the area law:

$$S_A \sim C \times \text{Area of boundary}$$

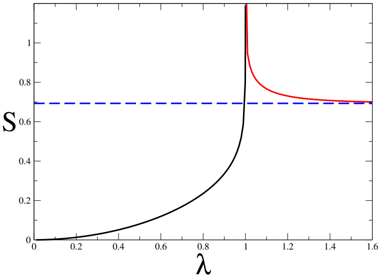
  where D is the dimensionality of space. The constant C is $\propto 1/(\text{lattice spacing})^{D-1}$ and is non-universal in general.
- entanglement occurs only near the boundary

One dimension

- when $D = 1$ something interesting happens: the constant is proportional to a logarithm

$$S_A \sim C \log(\text{correlation length } \xi)$$

- now the constant C is dimensionless and *universal*
- at a quantum critical point ξ diverges and so does the entanglement entropy



Measuring Entanglement

- measuring entropy of many-body systems is conceptually difficult: even at finite temperature we do it by integrating the specific heat
- however the situation is better for the Rényi entropies

$$S_A{}^{(n)} \propto -\log \operatorname{Tr} \rho_A{}^n = -\log \sum_j c_j^{2n}$$

- to simplify the discussion assume $n = 2$ and consider two independent identical copies of the whole system, so the composite system is in the state

$$|\Psi\rangle_1 |\Psi\rangle_2 = \sum_{j_1} \sum_{j_2} c_{j_1} c_{j_2} |\psi_{j_1}\rangle_{A1} |\psi_{j_1}\rangle_{B1} |\psi_{j_2}\rangle_{A2} |\psi_{j_2}\rangle_{B2}$$

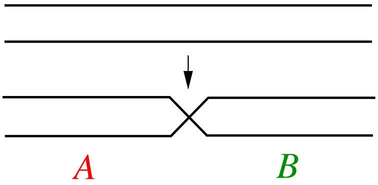
- let $\mathfrak{S}$ be a 'swap' operator which interchanges the states in A1 with those in A2 but leaves states in B1 and B2 the same:

$$\mathfrak{S}\,|\psi_{j_1}\rangle_{A1}|\psi_{j_1}\rangle_{B1}|\psi_{j_2}\rangle_{A2}|\psi_{j_2}\rangle_{B2} = |\psi_{j_2}\rangle_{A1}|\psi_{j_1}\rangle_{B1}|\psi_{j_1}\rangle_{A2}|\psi_{j_2}\rangle_{B2}$$

- then

$$\left({}_1\langle\Psi|{}_2\langle\Psi|\right)\mathfrak{S}\left(|\Psi\rangle_1|\Psi\rangle_2\right) = \sum_j c_j^4 = \operatorname{Tr}\rho_A{}^2$$

- on a system with local interactions, $\mathfrak{S}$ can be implemented locally as a quantum switch: eg a 1D quantum spin chain:



- initially the two decoupled chains have a hamiltonian $H = H_1 + H_2$ with a ground state $|0\rangle = |\Psi\rangle_1 |\Psi\rangle_2$
- after the switch, the hamiltonian is $H' = \mathfrak{S} H \mathfrak{S}^{-1}$, with a ground state $|0\rangle' = \mathfrak{S}|0\rangle$ with the *same* energy
- we need to measure the overlap

$$M = \langle 0|0\rangle' = \operatorname{Tr} \rho_A^2$$

- two proposals for how to do this:
- [5]
  - prepare the system in ground state $|0\rangle$ of H
  - flip the switch so the new hamiltonian is H'
  - the system finds itself in a higher energy state than $|0\rangle'$ and decays to this eg by emission of quasiparticles
  - decay rate $\propto |M|^2$
- [6]
  - introduce tunnelling between $|0\rangle$ and $|0\rangle'$, equivalent to adding a term $\propto \mathfrak{S}$ to the hamiltonian
  - the tunnelling amplitude is $\propto M$
  - this can be detected by preparing in one state and observing Rabi oscillations

[5] JC, Phys. Rev. Lett. **106**, 150404, 2011

[6] Abanin DA and Demler E, Phys. Rev. Lett. **109**, 020504, 2012

Summary

- entanglement is a fascinating and useful property of quantum mechanics
- entropy is a useful measure of entanglement for characterising many-body ground states (and also in quantum information theory)
- in principle it can be measured in condensed matter or cold atom experiments

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- entanglement is a fascinating and useful property of quantum mechanics
- entropy is a useful measure of entanglement for characterising many-body ground states (and also in quantum information theory)
- in principle it can be measured in condensed matter or cold atom experiments
- although the theory tells us a lot *die Alten* still have many secrets!