MEAN NUMBER OF CROSSING CLUSTERS
and related results

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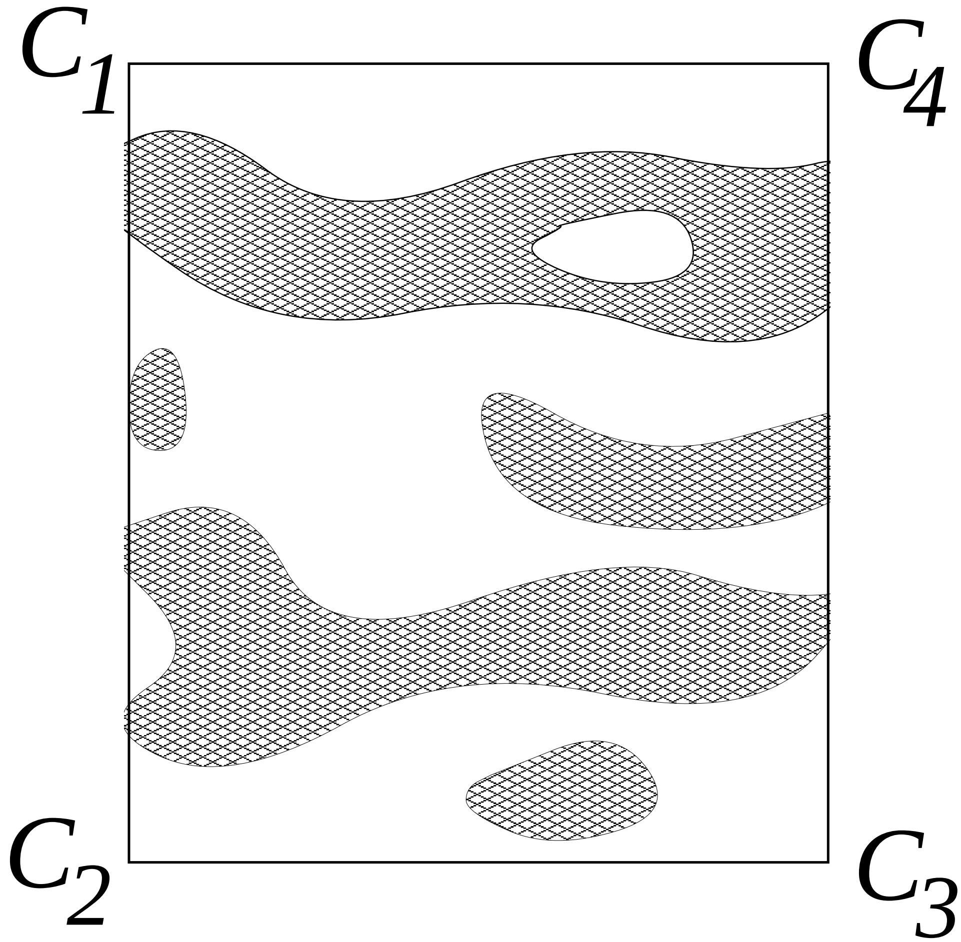
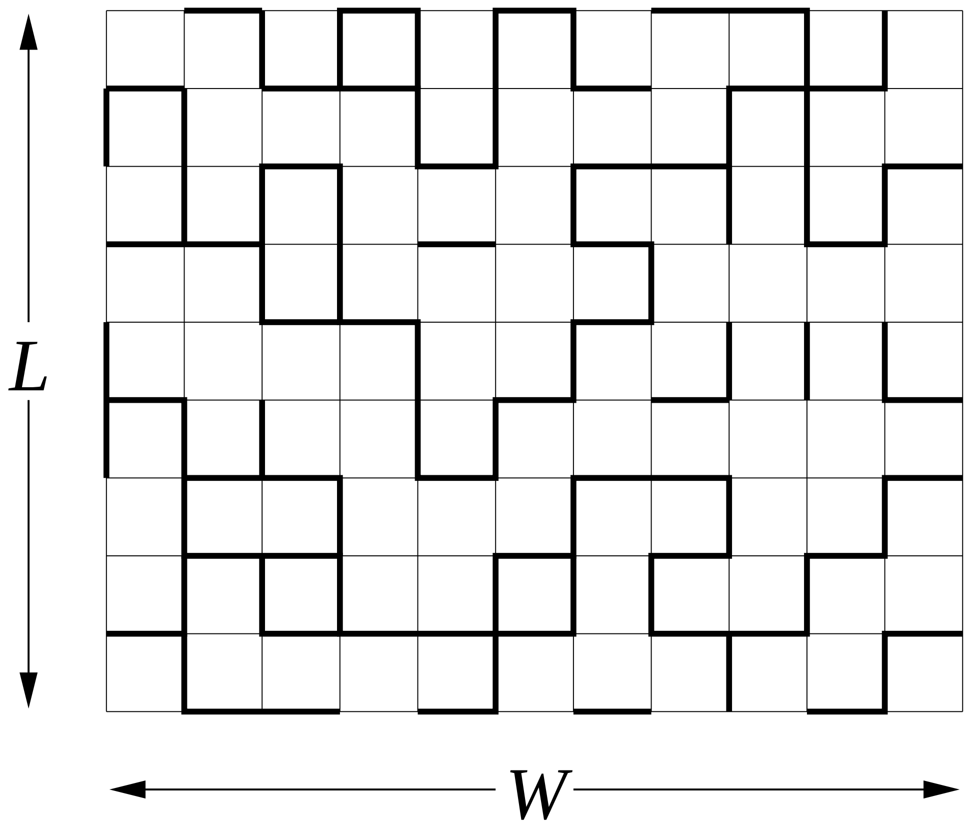
Outline.

- Crossing clusters and Potts model partition functions.
- Conformal field theory predictions: differential equations.
- Conformal field theory predictions: Coulomb gas methods.



Bond percolation: random spatial process in which edges of a graph (a regular lattice) are independently *open* (with prob. p) or *closed* $(1-p)$. Vertices connected by open edges form *clusters*; we study the connectivity properties of these clusters.

• focus on macroscopic geometrical objects: consider a critical percolation process on some finite but large lattice. N_c = number of distinct clusters which connect disjoint segments C_1C_2 and C_3C_4 of the boundary.



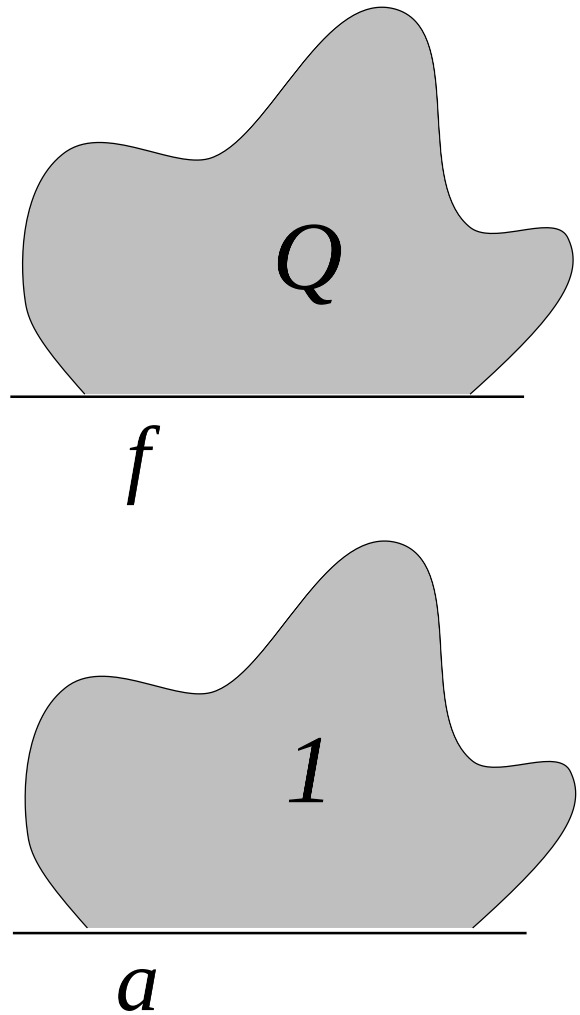
Percolation and the Potts model

'Spins' $s(r) = 1, 2, \ldots, a, \ldots, Q$ at vertices of the lattice.

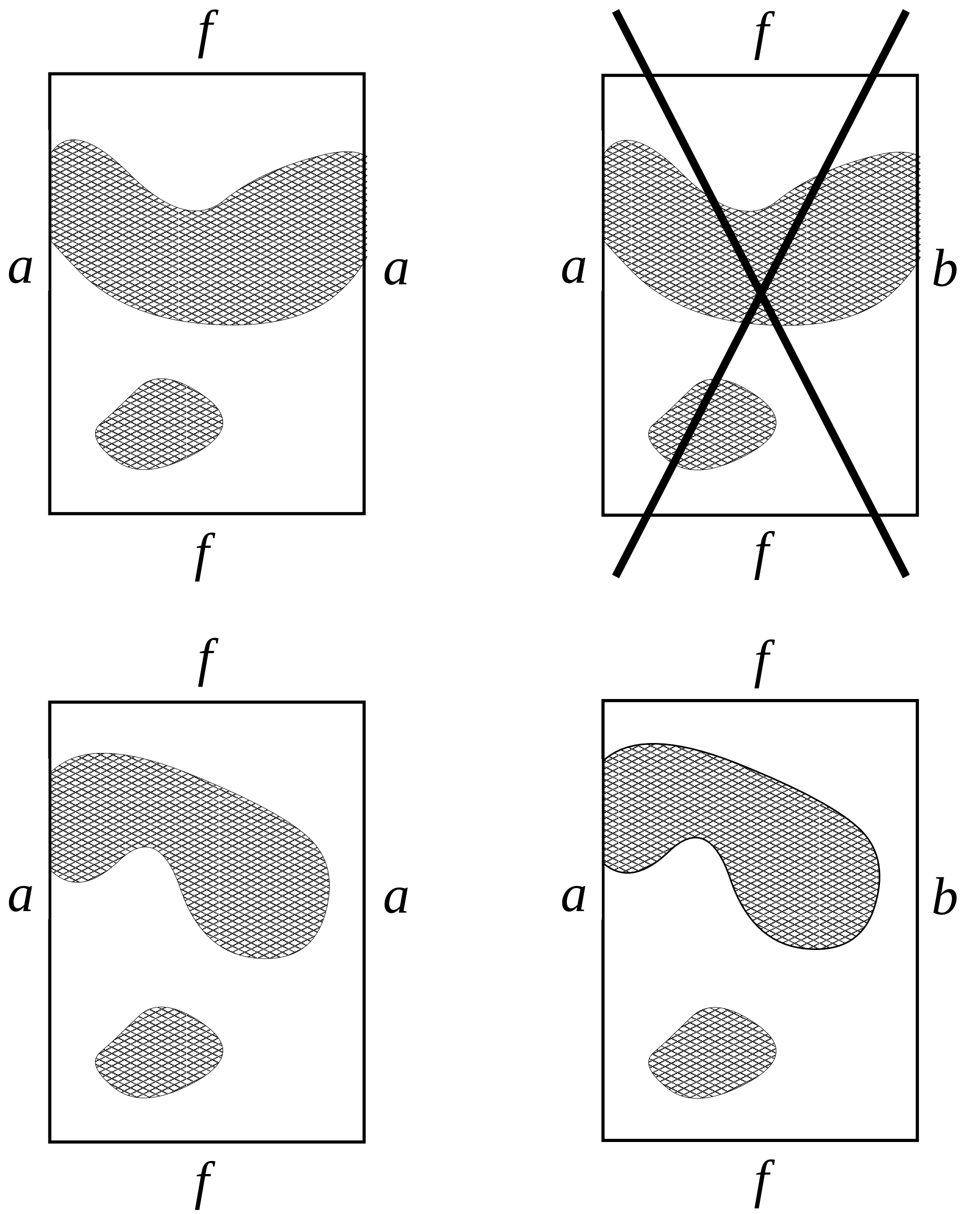
Partition function

$$\begin{aligned} Z &= \mathrm{Tr}\, e^{J\sum_{r,r'}\delta_{s(r),s(r')}} \\ &\propto \mathrm{Tr} \prod_{r,r'} \left[(1-p) + p\delta_{s(r),s(r')}\right] \qquad \text{where } p = 1 - e^{-J} \\ &= \sum_{\text{cluster configs}} p^{n_{\text{open}}}(1-p)^{n_{\text{closed}}}\, Q^{N_{\text{clusters}}} \qquad (*) \end{aligned}$$

$Q = 1 \Leftrightarrow$ percolation, but (*) can be continued to arbitrary Q.

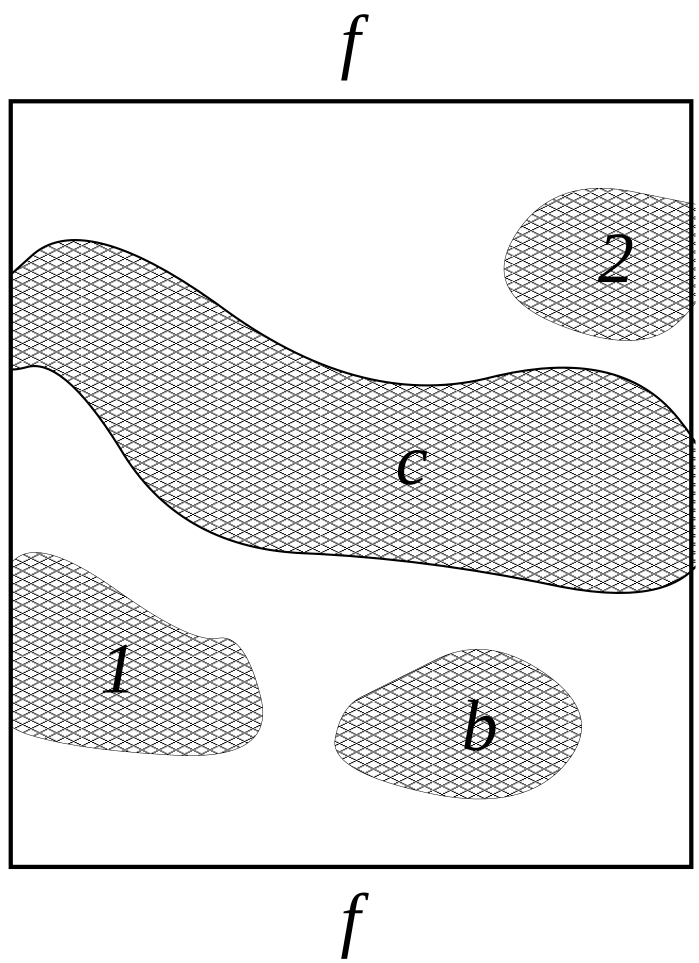


Boundary conditions: Potts spins *free* (f), or *fixed* into a given state (a): clusters which touch such a boundary are counted with weight 1 rather than Q.



Crossing probability:

$$Pr[N_c > 0] = Z_{aa}(Q = 1) - Z_{ab}(Q = 1)$$

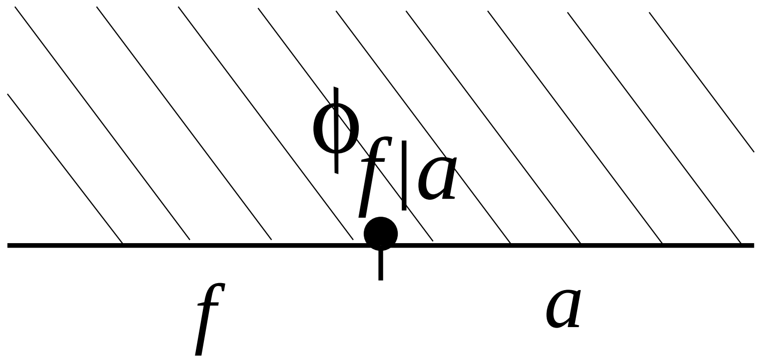


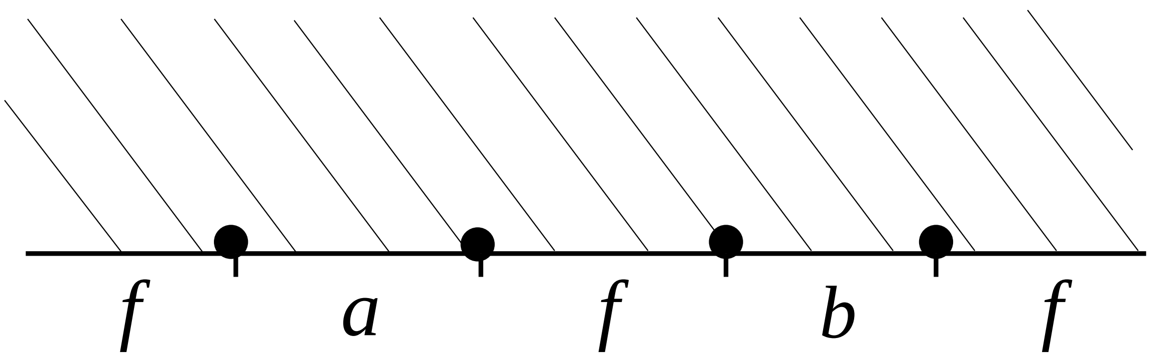
Mean number of crossing clusters:

$$Z_{ff} = \langle Q^{N_c+N_1+N_2+N_b} \rangle \qquad Z_{aa} = \langle Q^{N_b} \rangle$$
$$Z_{af} = \langle Q^{N_2+N_b} \rangle \qquad Z_{fa} = \langle Q^{N_1+N_b} \rangle$$

so that

$$E[N_c] = (\partial/\partial Q)|_{Q=1}(Z_{ff}Z_{aa}/Z_{fa}Z_{af})$$





• In both cases, we need to consider partition functions for which the boundary conditions are different on different parts of the boundary. Think of these as insertions of *boundary condition-changing operators* in the ensemble with free boundary conditions.

E.g.

$$Z_{ab} = \langle \phi_{f|a}(1)\phi_{a|f}(2)\phi_{f|b}(3)\phi_{b|f}(4)\rangle \qquad (*)$$

These behave just like ordinary correlation functions of local events.

Conformal field theory

- assumes the existence of the continuum limit of correlation functions of $\delta^{-x_\phi}\phi(r)$ as lattice spacing $\delta \to 0$.
- asserts that this limit is conformally covariant: $\phi(z) \to |dz'/dz|^{x_\phi}\phi(z')$ under a conformal mapping $z \to z'$. [Note: $x_{\phi_{f|a}}(Q=1)=0$, so that percolation partition functions and therefore crossing probabilities are conformally *invariant*.]
- if, under the conformal mapping which maps interior of the region into the unit disc, $(C_1,\ldots,C_4)$ are mapped into $(z_1,\ldots,z_4)$, then (*) depends only on the cross-ratio $\eta \equiv z_{12}z_{34}/z_{13}z_{24}$.
- such correlation functions often satisfy linear PDEs, and for $Q=2,3$ these are known to be second order (hypergeometric) equations. [Comes from considering effect of the infinitesimal transformation

$$z \to z' = z + \epsilon(z-z_1)^{-1} \tag{1}$$

on the correlation function $\langle\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle$, and assuming that $\delta\phi(z_1) \propto \epsilon(\partial/\partial z_1)^2\phi(z_1)$.]

Results

$$
\begin{aligned}
Z_{aa} &\propto {}_2F_1(-4x, \tfrac{1}{3} - \tfrac{4}{3}x; \tfrac{2}{3} - \tfrac{8}{3}x; 1-\eta) \\
Z_{ab} &\propto (1-\eta)^{\frac{1}{3} + \frac{8}{3}x} {}_2F_1(\tfrac{1}{3} - \tfrac{4}{3}x, \tfrac{2}{3} + \tfrac{4}{3}x; \tfrac{4}{3} + \tfrac{8}{3}x; 1-\eta)
\end{aligned}
$$

where the constants are fixed by $Z_{aa} \sim Z_{ab} \to 1$ as $\eta \to 0$, and $x = x(Q)$.

So:

$$
\begin{aligned}
Pr(N_c > 0) &= 1 - \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}(1-\eta)^{\frac{1}{3}} {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; 1-\eta) \\
&= \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{\frac{1}{3}} {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; \eta);
\end{aligned}
$$

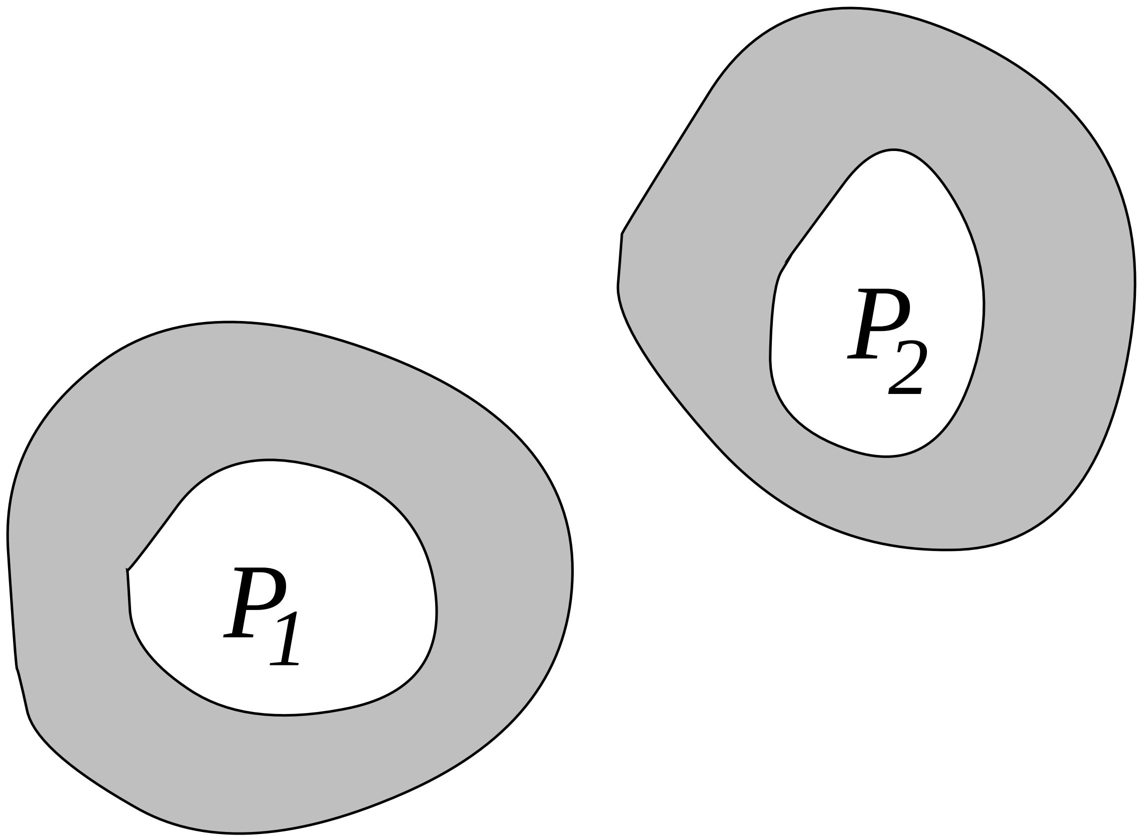
and

$$
\begin{aligned}
E[N_c] &= (\partial/\partial Q)|_{Q=1}(Z_{ff}Z_{aa}/Z_{fa}Z_{af}) \\
&= (\partial/\partial Q)|_{Q=1}(1-\eta)^{-2x(Q)}(1 + \tfrac{4\pi}{\sqrt{3}}x(Q)) \\
&\qquad \times {}_2F_1(-4x(Q), \tfrac{1}{3}; \tfrac{2}{3}; 1-\eta)
\end{aligned}
$$

which reduces to

$$
x'(1)\left[-2\ln(1-\eta) + \tfrac{4\pi}{\sqrt{3}} - 4\sum_{n=1}^{\infty} \frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3}+n)}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}+n)} \frac{(1-\eta)^n}{n}\right]
$$

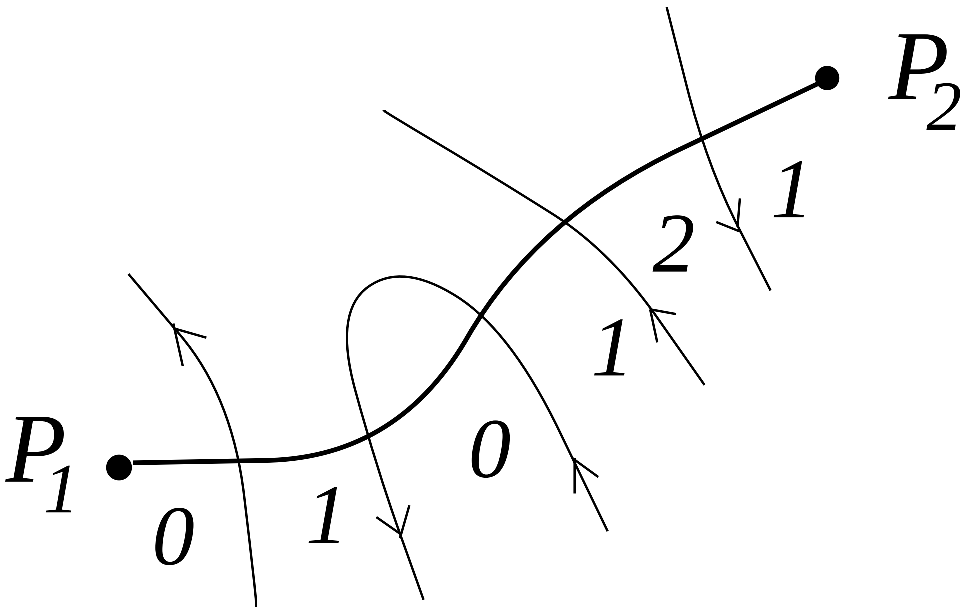
- 1/3 comes from the fact that at each regular singular point the indices are $(0,\alpha)$, and the sum of the indices $= 1$.
- $x'(1) = \sqrt{3}/8\pi$ may be found from $E[N_c] \sim Pr(N_c > 1)$ as $\eta \to 0$.
- for a $L \times W$ rectangle with $L \gg W$, $E[N_c] \sim 2x'(1)(L/W)$.
- partial results on the full probability distribution of N_c.



Coulomb gas methods

• work best for bulk quantities, e.g: how many clusters separate two points P_1 and P_2 of the plane ($= 2\times$ number of hulls)?

- Randomly assign an orientation to each loop, so each carries a total current ± 1. Current density $= J_\mu(r)$.
- define **height function** $h(2) - h(1) \equiv \int_1^2 J_\mu dS^\mu$, so $J_\mu = \epsilon_{\mu\nu}\partial^\nu h$.



- define **height function** $h(2)-h(1) \equiv \int_1^2 J_\mu dS^\mu$, so $J_\mu = \epsilon_{\mu\nu}\partial^\nu h$.
- In the continuum limit, $h(x)$ is conjectured to have a gaussian distribution

$$Pr(\{h\}) \propto \exp\big(-(k/2\pi)\int(\partial h)^2 d^2r\big).$$

Various methods then fix $k = 1/(2\sqrt{3}\pi)$.

- E[number of hulls threading P_1P_2] $= E[(h(2)-h(1))^2] \sim k \ln |P_1 - P_2|$
- conformally map to a cylinder: mean number of clusters wrapping round cylinder per unit length $= 1/(2\sqrt{3})$.

References.

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