Another Percolation Formula

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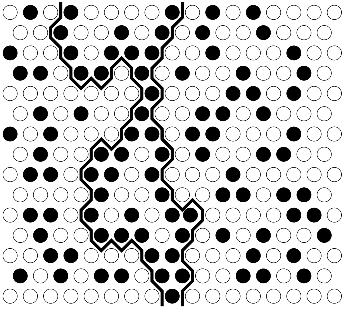
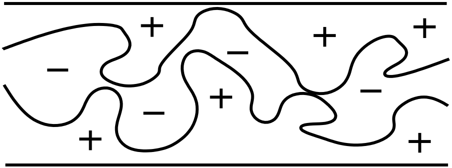
Joint work with Adam Gamsa

The Problem



- critical site percolation on triangular lattice in upper half-plane $\mathbf{H}$
- all sites on boundary are white, except:
- origin is black and is conditioned to be connected to infinity
- denote boundaries of white clusters connected to $\mathbf{R}-$ and $\mathbf{R}+$ by γ_- and γ_+
- what are the scaling limits of the probabilities that a given point $\zeta \in \mathbf{H}$ lies to the left of γ_-, to the right of γ_+, or in between?

Relation to quantum Hall physics

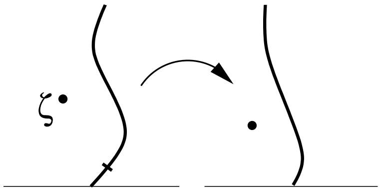


- conformally map half-plane to strip
- electron in strong magnetic field in random potential $V(r)$ approximately follows level lines $V(r) = E_F \sim$ boundaries of percolation clusters
- $V \to +\infty$ at edges
- electrons follow the boundary of cluster connected to the edges
- mean current density $\propto (d/dy)Pr(y \text{ lies above upper curve})$

Strategy

- single curve described by $\mathrm{SLE}_{\kappa=6}$
- $Pr(\text{point lies to L of curve})$ satisfies a simple ODE `[Schramm]`
- conformal field theory (CFT) interpretation
- generalisation to > 1 curve
- results
- ‘reverse engineer’ to get SLE description

Single curve – Schramm's formula



- what is the probability that $\zeta = u + iv$ lies to the left of the curve?
- γ described by SLE$_\kappa$ with $\kappa = 6$:

$$dg_t(z)/dt = 2/(g_t(z) - a_t) \qquad \text{with} \quad a_t = a_0 + \sqrt{\kappa}B_t$$

- $Pr(\zeta$ lies to L of SLE started at $a_0) =$
  $Pr(g_t(\zeta)$ lies to L of SLE started at $a_t)$

so

$$\left(\frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2} + \text{Re}\left[\frac{2}{\zeta - a_0}\frac{\partial}{\partial \zeta}\right]\right) P(\zeta; a_0) = 0$$

- P depends only on $t = (u - a_0)/v \Rightarrow$ 2nd order ODE
- boundary conditions $P \to 0$ as $t \to +\infty$, $P \to 1$ as $t \to -\infty$
- solution

$$P_{\text{left}} = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})}\, t\, {}_2F_1(\tfrac{1}{2}, \tfrac{4}{\kappa}; \tfrac{3}{2}; -t^2)$$

Relation to conformal field theory

$$P(\zeta; a_0) = \frac{\langle \mathcal{O}(\zeta)\phi_2(a_0)\phi_2(\infty)\rangle_{CFT}}{\langle \phi_2(a_0)\phi_2(\infty)\rangle_{CFT}}$$

where

- $\phi_2(x)$ conditions the partition function on a curve starting at boundary point x
- $\mathcal{O}(\zeta) = \mathbf{1}(\zeta \text{ to L of curve})$
- under $z \to z + 2\epsilon/(z - a_0)$

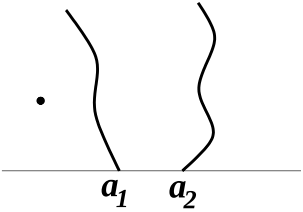
$$\phi_2(a_0) \to \phi_2(a_0) + 2\epsilon L_{-2}\phi_2(a_0)$$

- ϕ_2 is conjectured to have the special property that

$$L_{-2}\phi_2(a_0) = (\kappa/4)(\partial/\partial a_0)^2\phi_2(a_0)$$

- same differential equation

2-curve CFT calculation



$$P(\zeta; a_1, a_2) = \frac{\langle \mathcal{O}(\zeta)\phi_2(a_1)\phi_2(a_2)\phi_2(\infty)\phi_2(\infty)\rangle}{\langle \phi_2(a_1)\phi_2(a_2)\phi_2(\infty)\phi_2(\infty)\rangle}$$

- numerator N satisfies 2 equations:

$$\left(\frac{\kappa}{2}\frac{\partial^2}{\partial a_1^2} + \frac{2}{a_2 - a_1}\frac{\partial}{\partial a_2} - \frac{2h_2}{(a_2 - a_1)^2} + \mathrm{Re}\left[\frac{2}{\zeta - a_1}\frac{\partial}{\partial \zeta}\right]\right) N = 0$$

$$\left(\frac{\kappa}{2}\frac{\partial^2}{\partial a_2^2} + \frac{2}{a_1 - a_2}\frac{\partial}{\partial a_1} - \frac{2h_2}{(a_1 - a_2)^2} + \mathrm{Re}\left[\frac{2}{\zeta - a_2}\frac{\partial}{\partial \zeta}\right]\right) N = 0$$

where $h_2 = (6 - \kappa)/2\kappa$. (Denominator satisfies similar equations without last terms.)

Fusion rules

- as $\delta = a_2 - a_1 \to 0$

$$\phi_2(a_1) \cdot \phi_2(a_2) = \delta^{1-6/\kappa}\phi_1(a_1) + \delta^{2/\kappa}\phi_3(a_1)$$

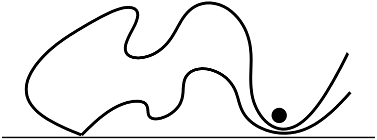
- any solution of these equations has the form

$$\delta^{1-6/\kappa}\big(F_1(a_1,\zeta) + O(\delta)\big) + \delta^{2/\kappa}\big(F_3(a_1,\zeta) + O(\delta)\big)$$

  where F_1 satisfies a 1st order PDE and F_3 a 3rd order PDE.
- conditioning curves to go to ∞ picks out F_3
- since P depends only on $t = u/v$ this leads to a 3rd order ODE
- one solution is $P =$ const. $\Rightarrow$ 2nd order Riemann equation for dP/dt
- general solution

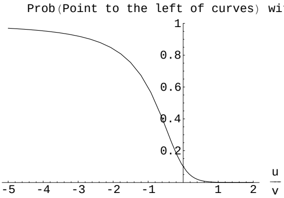
$$(1+t^2)^{1-\frac{8}{\kappa}}\left(A\,{}_2F_1(\tfrac{1}{2}+\tfrac{4}{\kappa},1-\tfrac{4}{\kappa};\tfrac{1}{2};-t^2) + B\,t\,{}_2F_1(1+\tfrac{4}{\kappa},\tfrac{3}{2}-\tfrac{4}{\kappa};\tfrac{3}{2};-t^2)\right)$$

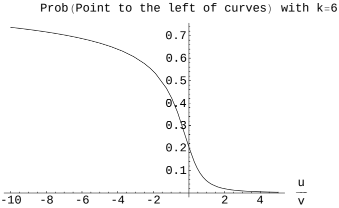
Boundary conditions

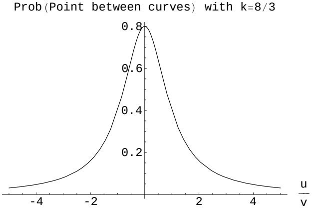


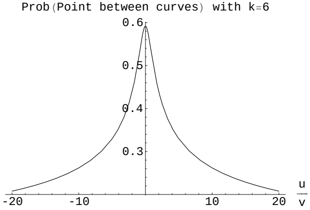
- as $t \to +\infty$, $P_{\text{left}} \sim t^{-x_4}$ where $x_4 = (24/\kappa) - 2$ is the boundary 4-leg exponent $\Rightarrow$ fixes B/A
- as $t \to -\infty$, $P_{\text{left}} \to 1 \Rightarrow$ fixes A
- $P_{\text{right}}(t) = P_{\text{left}}(-t)$
- $P_{\text{middle}}(t) = 1 - P_{\text{left}}(t) - P_{\text{right}}(t)$

Results


Prob(Point to the left of curves) with k=8/3
1
0.8
0.6
0.4
0.2
-5 -4 -3 -2 -1 1 2
u
v

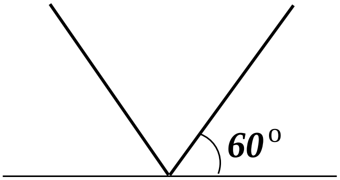

Prob(Point to the left of curves) with k=6
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-10 -8 -6 -4 -2 2 4
u
v


Prob(Point between curves) with k=8/3
0.8
0.6
0.4
0.2
-4 -2 2 4
u
v


Prob(Point between curves) with k=6
0.6
0.5
0.4
0.3
-20 -10 10 20
u
v

Extremal cases

- $\kappa = 0$



- $\kappa = 8$ (recall $P_{\text{left}} = \frac{1}{2}$ for 1 curve – space-filling)
- let $u/v = \tan\phi$

$$\begin{aligned} P_{\text{left}} &= \tfrac{1}{4}(1 - 2\phi/\pi) \\ P_{\text{middle}} &= \tfrac{1}{2} \\ P_{\text{right}} &= \tfrac{1}{4}(1 + 2\phi/\pi) \end{aligned}$$

Reverse engineering SLE(κ, ρ)

- first CFT equation corresponds to assuming that Loewner driving term satisfies

$$\begin{aligned} da_1 &= \sqrt{\kappa} dB_t + \frac{2dt}{a_1 - a_2} \\ da_2 &= \frac{2dt}{a_2 - a_1} \end{aligned}$$

- SLE$(\kappa, 2)$
- this generates the measure on curve #1 conditioned on the existence of curve #2
- similarly with $1 \leftrightarrow 2$
- follows from scaling and commutativity, but less trivial for > 2 curves

Multiple SLE

- but we could also take any linear combination $\sum_j b_j \mathcal{D}_j P = \ldots$ of the CFT equations
- corresponds to a Loewner map satisfying

$$\dot{g}_t(z) = \sum_j \frac{2b_j}{g_t(z) - a_{jt}}$$

where

$$da_j = \sqrt{b_j \kappa}\, dB_t^{(j)} + \sum_{k \neq j} \frac{2(b_j + b_k)dt}{a_j - a_k}$$

- corresponds to growing curves at different speeds
- if CFT correctly describes continuum limit of lattice models then different choices should give same joint measure on curves
- when the b_j are all equal this is Dyson's Brownian motion

Summary

- we have derived formulae for the expected values of simple observables of the conjectured scaling limit of 2 curves in lattice models like percolation
- generalization to N curves possible but requires solving an ODE of order $N-1$
- CFT suggests that the measure on a single curve is given by SLE$(\kappa, 2)$, and that joint measure on curves is given by ‘multiple SLE’