Another Percolation Formula

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The Problem

- critical site percolation on triangular lattice in upper half-plane $\mathbb{H}$
- all sites on boundary are white, except:
- origin is black and is conditioned to be connected to infinity
- denote boundaries of white clusters connected to $\mathbb{R}^-$ and $\mathbb{R}^+$ by $\gamma_-$ and $\gamma_+$
- what are the scaling limits of the probabilities that a given point $\zeta \in \mathbb{H}$ lies to the left of $\gamma_-$, to the right of $\gamma_+$, or in between?
Relation to quantum Hall physics

- conformally map half-plane to strip
- electron in strong magnetic field in random potential $V(r)$ approximately follows level lines $V(r) = E_F \sim$ boundaries of percolation clusters
- $V \rightarrow +\infty$ at edges
- electrons follow the boundary of cluster connected to the edges
- mean current density $\propto (d/dy)Pr(y \text{ lies above upper curve})$
Strategy

- single curve described by $\text{SLE}_{\kappa=6}$
- $Pr(\text{point lies to L of curve})$ satisfies a simple ODE [Schramm]
- conformal field theory (CFT) interpretation
- generalisation to $>1$ curve
- results
- ‘reverse engineer’ to get SLE description
what is the probability that $\zeta = u + iv$ lies to the left of the curve?

$\gamma$ described by SLE$_\kappa$ with $\kappa = 6$:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

with $$a_t = a_0 + \sqrt{\kappa}B_t$$

$Pr(\zeta \text{ lies to L of SLE started at } a_0) = Pr(g_t(\zeta) \text{ lies to L of SLE started at } a_t)$

so

$$\left( \frac{\kappa}{2} \frac{\partial^2}{\partial a_0^2} + \Re \left[ \frac{2}{\zeta - a_0} \frac{\partial}{\partial \zeta} \right] \right) P(\zeta; a_0) = 0$$
\( P \) depends only on \( t = (u - a_0)/v \Rightarrow 2\text{nd order ODE} \)

- boundary conditions: \( P \to 0 \) as \( t \to +\infty \), \( P \to 1 \) as \( t \to -\infty \)

- solution

\[
P_{\text{left}} = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi} \Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \, t \, _2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; -t^2\right)
\]
Relation to conformal field theory

\[ P(\zeta; a_0) = \frac{\langle O(\zeta)\phi_2(a_0)\phi_2(\infty) \rangle_{CFT}}{\langle \phi_2(a_0)\phi_2(\infty) \rangle_{CFT}} \]

where

- \( \phi_2(x) \) conditions the partition function on a curve starting at boundary point \( x \)
- \( O(\zeta) = 1(\zeta \text{ to } L \text{ of curve}) \)
- under \( z \rightarrow z + 2\epsilon/(z - a_0) \)

\[ \phi_2(a_0) \rightarrow \phi_2(a_0) + 2\epsilon L_{-2}\phi_2(a_0) \]

- \( \phi_2 \) is conjectured to have the special property that

\[ L_{-2}\phi_2(a_0) = (\kappa/4)(\partial/\partial a_0)^2 \phi_2(a_0) \]

- same differential equation
2-curve CFT calculation

\[ P(\zeta; a_1, a_2) = \frac{\langle \mathcal{O}(\zeta) \phi_2(a_1) \phi_2(a_2) \phi_2(\infty) \phi_2(\infty) \rangle}{\langle \phi_2(a_1) \phi_2(a_2) \phi_2(\infty) \phi_2(\infty) \rangle} \]

- numerator \( N \) satisfies 2 equations:

\[
\begin{align*}
\left( \frac{\kappa}{2} \frac{\partial^2}{\partial a_1^2} + \frac{2}{a_2 - a_1} \frac{\partial}{\partial a_2} - \frac{2h_2}{(a_2 - a_1)^2} + \text{Re} \left[ \frac{2}{\zeta - a_1} \frac{\partial}{\partial \zeta} \right] \right) N &= 0 \\
\left( \frac{\kappa}{2} \frac{\partial^2}{\partial a_2^2} + \frac{2}{a_1 - a_2} \frac{\partial}{\partial a_1} - \frac{2h_2}{(a_1 - a_2)^2} + \text{Re} \left[ \frac{2}{\zeta - a_2} \frac{\partial}{\partial \zeta} \right] \right) N &= 0
\end{align*}
\]

where \( h_2 = (6 - \kappa)/2\kappa \). (Denominator satisfies similar equations without last terms.)
Fusion rules

- as $\delta = a_2 - a_1 \to 0$

$$\phi_2(a_1) \cdot \phi_2(a_2) = \delta^{1-6/\kappa} \phi_1(a_1) + \delta^{2/\kappa} \phi_3(a_1)$$

- any solution of these equations has the form

$$\delta^{1-6/\kappa} (F_1(a_1, \zeta) + O(\delta)) + \delta^{2/\kappa} (F_3(a_1, \zeta) + O(\delta))$$

where $F_1$ satisfies a 1st order PDE and $F_3$ a 3rd order PDE.

- conditioning curves to go to $\infty$ picks out $F_3$

- since $P$ depends only on $t = u/v$ this leads to a 3rd order ODE

- one solution is $P = \text{const.} \Rightarrow 2$nd order Riemann equation for $dP/dt$

- general solution

$$(1+t^2)^{1-8/\kappa} (A_2F_1(\frac{1}{2} + \frac{4}{\kappa}, 1 - \frac{4}{\kappa}; \frac{1}{2}; -t^2) + B t_2F_1(1 + \frac{4}{\kappa}, \frac{3}{2} - \frac{4}{\kappa}; \frac{3}{2}; -t^2))$$
Boundary conditions

- as $t \to +\infty$, $P_{\text{left}} \sim t^{-x_4}$ where $x_4 = (24/\kappa) - 2$ is the boundary 4-leg exponent $\Rightarrow$ fixes $B/A$
- as $t \to -\infty$, $P_{\text{left}} \to 1 \Rightarrow$ fixes $A$
- $P_{\text{right}}(t) = P_{\text{left}}(-t)$
- $P_{\text{middle}}(t) = 1 - P_{\text{left}}(t) - P_{\text{right}}(t)$
Results

\[
\text{Prob(Point to the left of curves) with } k=8/3
\]

\[
\text{Prob(Point between curves) with } k=8/3
\]

\[
\text{Prob(Point to the left of curves) with } k=6
\]

\[
\text{Prob(Point between curves) with } k=6
\]
Extremal cases

- $\kappa = 0$

- $\kappa = 8$ (recall $P_{\text{left}} = \frac{1}{2}$ for 1 curve – space-filling)

- let $u/v = \tan \phi$

\[
\begin{align*}
P_{\text{left}} &= \frac{1}{4}(1 - 2\phi/\pi) \\
P_{\text{middle}} &= \frac{1}{2} \\
P_{\text{right}} &= \frac{1}{4}(1 + 2\phi/\pi)
\end{align*}
\]
Reverse engineering $\text{SLE}(\kappa, \rho)$

- first CFT equation corresponds to assuming that Loewner driving term satisfies

\[
\begin{align*}
da_1 &= \sqrt{\kappa} dB_t + \frac{2dt}{a_1 - a_2} \\
da_2 &= \frac{2dt}{a_2 - a_1}
\end{align*}
\]

- $\text{SLE}(\kappa, 2)$
- this generates the measure on curve #1 conditioned on the existence of curve #2
- similarly with $1 \leftrightarrow 2$
- follows from scaling and commutativity, but less trivial for $> 2$ curves
Multiple SLE

- but we could also take any linear combination $\sum_j b_j D_j P = \ldots$ of the CFT equations
- corresponds to a Loewner map satisfying

$$\dot{g}_t(z) = \sum_j \frac{2b_j}{g_t(z) - a_{jt}}$$

where

$$da_j = \sqrt{b_j \kappa} dB_t^{(j)} + \sum_{k \neq j} \frac{2(b_j + b_k)dt}{a_j - a_k}$$

- corresponds to growing curves at different speeds
- if CFT correctly describes continuum limit of lattice models then different choices should give same joint measure on curves
- when the $b_j$ are all equal this is Dyson’s Brownian motion
we have derived formulae for the expected values of simple observables of the conjectured scaling limit of 2 curves in lattice models like percolation

generalization to $N$ curves possible but requires solving an ODE of order $N - 1$

CFT suggests that the measure on a single curve is given by $\text{SLE}(\kappa, 2)$, and that joint measure on curves is given by ‘multiple SLE’