# Entanglement in Quantum Field Theory

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## Outline

- Quantum entanglement in general and its quantification
- Path integral approach
- Entanglement entropy in 1+1-dimensional CFT
- Higher dimensions
- Mixed states and negativity

Work largely carried out with Pasquale Calabrese (Pisa) and Erik Tonni (Trieste)

## Quantum Entanglement (Bipartite, Pure State)

- quantum system in a pure state  $|\Psi\rangle$ , density matrix  $\rho = |\Psi\rangle\langle\Psi|$
- $\blacktriangleright \mathcal{H} = \mathcal{H}_{\mathbf{A}} \otimes \mathcal{H}_{\mathbf{B}}$
- Alice can make unitary transformations and measurements only in A, Bob only in the complement B
- in general Alice's measurements are entangled with those of Bob
- example: two spin-<sup>1</sup>/<sub>2</sub> degrees of freedom

$$|\psi\rangle = \cos\theta |\uparrow\rangle_{\mathbf{A}} |\downarrow\rangle_{\mathbf{B}} + \sin\theta |\downarrow\rangle_{\mathbf{A}} |\uparrow\rangle_{\mathbf{B}}$$

Measuring bipartite entanglement in pure states

Schmidt decomposition:

$$|\Psi
angle = \sum_{j} c_{j} |\psi_{j}
angle_{\mathcal{A}} \otimes |\psi_{j}
angle_{\mathcal{B}}$$

with  $c_j \ge 0$ ,  $\sum_j c_j^2 = 1$ , and  $|\psi_j\rangle_A$ ,  $|\psi_j\rangle_B$  orthonormal.

- one quantifier of the amount of entanglement is the entropy  $S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$
- if  $c_1 = 1$ , rest zero, S = 0 and  $|\Psi\rangle$  is unentangled
- ► if all c<sub>j</sub> equal, S ~ log min(dimH<sub>A</sub>, dimH<sub>B</sub>) maximal entanglement

► equivalently, in terms of Alice's reduced density matrix:  $\rho_A \equiv \text{Tr}_B |\Psi\rangle\langle\Psi|$ 

$$S_A = -\text{Tr}_A \rho_A \log \rho_A = S_B$$
 von Neumann entropy

similar information is contained in the Rényi entropies

$$S_{\mathcal{A}}^{(n)} = (1-n)^{-1} \log \operatorname{Tr}_{\mathcal{A}} \rho_{\mathcal{A}}^{n}$$

$$\blacktriangleright S_{\mathbf{A}} = \lim_{n \to 1} S_{\mathbf{A}}^{(n)}$$

- other measures of entanglement exist, but entropy has several nice properties: additivity, convexity, ...
- it is monotonic under Local Operations and Classical Communication (LOCC)
- it gives the amount of classical information required to specify ρ<sub>A</sub> (important for numerical computations)
- it gives a basis-independent way of identifying and characterising quantum phase transitions
- in a relativistic QFT the entanglement in the vacuum encodes all the data of the theory (spectrum, anomalous dimensions, ...)

# Entanglement entropy in QFT

In this talk we consider the case when:

- ► the degrees of freedom are those of a local relativistic QFT in large region R in R<sup>d</sup>
- the whole system is in the vacuum state  $|0\rangle$
- ► A is the set of degrees of freedom in some large (compact) subset of R, so we can decompose the Hilbert space as

$$\mathcal{H} = \mathcal{H}_{A} \otimes \mathcal{H}_{B}$$

- in fact this makes sense only in a cut-off QFT (e.g. a lattice), and some of the results will in fact be cut-off dependent
- How does S<sub>A</sub> depend on the size and geometry of A and the universal data of the QFT?

#### Rényi entropies from the path integral (d = 1)



wave functional Ψ({a}, {b}) ∝ conditioned path integral in imaginary time from τ = −∞ to τ = 0:

$$\Psi(\{a\},\{b\}) = Z_1^{-1/2} \int_{a(0)=a,b(0)=b} [da(\tau)] [db(\tau)] e^{-(1/\hbar)S[\{a(\tau)\},\{b(\tau)\}]}$$

where  $S = \int_{-\infty}^{0} L(a(\tau), b(\tau)) d\tau$ 

 similarly Ψ\*({a}, {b}) is given by the path integral from τ = 0 to +∞

$$\rho_{A}(\boldsymbol{a}_{1},\boldsymbol{a}_{2}) = \int db \,\Psi(\boldsymbol{a}_{1},\boldsymbol{b}) \Psi^{*}(\boldsymbol{a}_{2},\boldsymbol{b})$$

this is given by the path integral over ℝ<sup>2</sup> cut open along A ∩ {τ = 0}, divided by Z<sub>1</sub>:



#### Rényi entropies: example n = 2

$$\rho_{\mathsf{A}}(\mathbf{a}_1,\mathbf{a}_2) = \int db \,\Psi(\mathbf{a}_1,b) \Psi^*(\mathbf{a}_2,b)$$

 $\operatorname{Tr}_{A} \rho_{A}^{2} = \int da_{1} da_{2} db_{1} db_{2} \Psi(a_{1}, b_{1}) \Psi^{*}(a_{2}, b_{1}) \Psi(a_{2}, b_{2}) \Psi^{*}(a_{1}, b_{2})$ 



$$\mathrm{Tr}_{\mathbf{A}} \rho_{\mathbf{A}}^{2} = Z(\mathcal{R}_{2})/Z_{1}^{2}$$

where  $Z(\mathcal{R}_2)$  is the euclidean path integral (partition function) on an 2-sheeted conifold  $\mathcal{R}_2$ 

#### ► in general

$$\operatorname{Tr}_{\mathbf{A}} \rho_{\mathbf{A}}{}^{n} = Z(\mathcal{R}_{n})/Z_{1}^{n}$$

where the half-spaces are connected as



to form  $\mathcal{R}_n$ .

conical singularity of opening angle 2πn at the boundary of
 A and B on τ = 0

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	и		v	

▶ if space is 1d and A is an interval (u, v) (and B is the complement) then Z(R<sub>n</sub>) can be thought of as the insertion of twist operators into n copies of the CFT:

$$Z(\mathcal{R}_n)/Z_1^n = \langle \mathcal{T}_{-n}(u)\mathcal{T}_n(v) \rangle_{(CFT)^n}$$

- these have similar properties to other local operators e.g.
- in a massless QFT (a CFT)

$$\langle \mathcal{T}_{-n}(u)\mathcal{T}_{n}(v)\rangle \sim |u-v|^{-2\Delta_{n}}$$

▶ in a massive QFT,

$$\langle \mathcal{T}_n 
angle \sim m^{\Delta_n}$$
 and  $\langle \mathcal{T}_{-n}(u) \mathcal{T}_n(v) 
angle - \langle \mathcal{T}_n 
angle^2 \sim \mathrm{e}^{-2m|u-v|}$ 

► main result for d = 1: ∆<sub>n</sub> = (c/12)(n - 1/n) where c is the central charge of the UV CFT





- consider a cone of radius *R* and opening angle  $\alpha = 2\pi n$
- w = log z maps this into a cylinder of length log R and circumference α

$$\frac{Z_{\text{cone}}(2\pi n)}{Z_{\text{cone}}(2\pi)^n} = \frac{Z_{\text{cyl}}(2\pi n)}{Z_{\text{cyl}}(2\pi)^n} \sim \frac{e^{\pi c \log R/12\pi n}}{(e^{\pi c \log R/12\pi})^n} \sim R^{-\Delta_n}$$

From this we see for example that for a single interval A of length ℓ [Holzhey, Larsen, Wilczek 1994]

$$S_{\mathbf{A}} \sim -\left. \frac{\partial}{\partial n} \right|_{n=1} \, \ell^{-2\Delta_n} = (\mathbf{C}/\mathbf{3}) \log(\ell/\epsilon)$$

- ► note this is much less than the entanglement in a *typical* state which is O(ℓ)
- many more universal results, eg finite-temperature cross-over between entanglement and thermodynamic entropy (β = 1/k<sub>B</sub>T):

$$egin{array}{rcl} S_{m{A}} &=& (c/3)\log\left((eta/\pi)\sinh(\pi\ell/eta)
ight) \ &\sim& (c/3)\log\ell & ext{for }\ell\lleta \ &\sim& \pi c\ell/3eta & ext{for }\ell\ggeta \end{array}$$

#### Massive QFT in 1+1 dimensions

▶ for 2 intervals 
$$A = (-\infty, 0)$$
 and  $B = (0, \infty)$ 

$$S_A \sim -(c/6) \log(m/\epsilon)$$
 as  $m \to 0$ 

- the entanglement diverges at a quantum phase transition and gives a basis-independent way of characterising the underlying CFT
- this is numerically the most accurate way of determining c for a given lattice model



$$Z(\mathcal{R}_n)/Z^n = \langle \mathcal{T}_{-n}(u_1)\mathcal{T}_{n}(v_1)\mathcal{T}_{-n}(u_2)\mathcal{T}_{n}(v_2)\rangle$$

in general there is no simple result but for |u<sub>j</sub> − v<sub>j</sub>| ≪ |u<sub>1</sub> − u<sub>2</sub>| we can use an operator product expansion

$$\mathcal{T}_{-n}(u)\mathcal{T}_{n}(v) = \sum_{\{k_{j}\}} C_{\{k_{j}\}}(u-v) \prod_{j=1}^{n} \Phi_{k_{j}}(\frac{1}{2}(u+v)_{j})$$

in terms of a complete set of local operators  $\Phi_{k_i}$ 

this gives the Rényi entropies as an expansion

$$\sum_{\{k_j\}} C_{\{k_j\}}^2 \eta^{\sum_j \Delta_{k_j}} \text{ where } \eta = ((u_1 - v_1)(u_2 - v_2))/((u_1 - u_2)(v_1 - v_2))$$

► the C<sub>{k<sub>i</sub></sub>} encode all the data of the CFT

#### Higher dimensions d > 1



• the conifold  $\mathcal{R}_n$  is now  $\{2d \text{ conifold}\} \times \{\text{boundary } \partial A\}$ 

$$\log Z(\mathcal{R}_n) \sim \operatorname{Vol}(\partial A) \cdot \epsilon^{-(d-1)}$$

this is the 'area law' in 3+1 dimensions [Srednicki 1992]

coefficient is non-universal

for even d + 1 there are interesting corrections to the area law

 $\operatorname{Vol}(\partial A) m^2 \log(m\epsilon), \quad a \log(R_A/\epsilon)$ 

whose coefficients are related to curvature anomalies of the CFT and are universal

- e.g. a is the 'a-anomaly' which is supposed to decrease along RG flows between CFTs [Komargodski-Schwimmer]
- it would interesting to give an entanglement-based argument for this result [Casini-Huerta]

# Mutual Information of multiple regions



- ► the non-universal 'area' terms cancel in  $I^{(n)}(A_1, A_2) = S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1 \cup A_2}$
- this mutual Rényi information is expected to be universal depending only on the geometry and the data of the CFT
- e.g. for a free scalar field in 3+1 dimensions [JC 2013]

$$I^{(n)}(A_1, A_2) \sim \frac{n^4 - 1}{15n^3(n-1)} \left(\frac{R_1R_2}{r_{12}^2}\right)^2$$

# Negativity

- however, mutual information does not correctly capture the quantum entanglement between A<sub>1</sub> and A<sub>2</sub>, e.g. it also includes classical correlations at finite temperature
- more generally we want a way of quantifying entanglement in a mixed state ρ<sub>A1∪A2</sub>
- one computable measure is negativity [Vidal, Werner 2002]
- let  $\rho_{A_1 \cup A_2}^{T_2}$  be the *partial* transpose:

$$\rho_{A_1 \cup A_2}^{T_2}(a_1, a_2; a_1', a_2') = \rho_{A_1 \cup A_2}(a_1, a_2'; a_1', a_2)$$

► Tr  $\rho_{A_1 \cup A_2}^{T_2} = 1$ , but it may now have negative eigenvalues  $\lambda_k$ Log-negativity  $\mathcal{N} = \log \operatorname{Tr} \left| \rho_{A_1 \cup A_2}^{T_2} \right| = \log \sum_k |\lambda_k|$ 

 if this is > 0 there are negative eigenvalues. This is an entanglement measure with nice properties, including being an LOCC monotone

# Negativity in QFT

'replica trick'

$$\operatorname{Tr}(\rho_{A_1\cup A_2}^{T_2})^n = \sum_k \lambda_k^n = \sum_k |\lambda_k|^n \quad \text{if } n \text{ is even}$$

- analytically continue to n = 1 to get  $\sum_{k} |\lambda_k|$  (!!)
- we can compute  $\operatorname{Tr}(\rho_{A_1 \cup A_2}^{T_2})^n$  by connecting the half-spaces in the opposite order along  $A_2$ :





• for  $\rho_{A_1 \cup A_2}$  we need  $\langle \mathcal{T}_{-n}(u_1) \mathcal{T}_{n}(v_1) \mathcal{T}_{-n}(u_2) \mathcal{T}_{n}(v_2) \rangle$ 

• for 
$$\rho_{A_1 \cup A_2}^{T_2}$$
 we need  $\langle \mathcal{T}_{-n}(u_1) \mathcal{T}_n(v_1) \mathcal{T}_n(u_2) \mathcal{T}_{-n}(v_2) \rangle$   
But

$$\begin{array}{rcl} \mathcal{T}_n \cdot \mathcal{T}_n &\cong& \mathcal{T}_n & (n \text{ odd}) & \to 1 & \text{for } n \to 1 \\ &\cong& \mathcal{T}_{n/2} \otimes \mathcal{T}_{n/2} & (n \text{ even}) \to \mathcal{T}_{1/2} \otimes \mathcal{T}_{1/2} & \text{for } n \to 1 \end{array}$$

so we get a non-trivial result if the intervals are close



so for example for d > 1 for 2 large regions a finite distance apart

 $\mathcal{N}(\textit{A}_{1},\textit{A}_{2}) \propto$  Area of common boundary between  $\textit{A}_{1}$  and  $\textit{A}_{2}$ 

 N appears to decay exponentially with separation of the regions, even in a CFT

# Other Related Interesting Stuff

- topological phases in 2 (and higher) spatial dimensions entanglement entropy distinguishes these in absence of local order parameter [Kitaev/Preskill and many others]
- entanglement spectrum' of the eigenvalues of log ρ<sub>A</sub> [Haldane]
- Shannon entropy -Tr|Ψ|<sup>2</sup> log |Ψ<sup>2</sup>| seems to have interesting properties depute being basis-dependent
- holographic computation of entanglement using AdS/CFT [Ryu/Takayanagi and many others]
- time-dependence in particular quantum quenches where the system is prepared in a state |ψ⟩ which is not an eigenstate of hamiltonian: how do entanglement (and correlation functions) behave?