Entanglement in Quantum Field Theory

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Outline

- Quantum entanglement in general and its quantification
- Path integral approach
- Entanglement entropy in 1+1-dimensional CFT
- Higher dimensions
- Mixed states and negativity

Work largely carried out with Pasquale Calabrese (Pisa)
and Erik Tonni (Trieste)

Quantum Entanglement (Bipartite, Pure State)

- quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$
- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Alice can make unitary transformations and measurements only in A, Bob only in the complement B
- in general Alice's measurements are entangled with those of Bob
- example: two spin-$\frac{1}{2}$ degrees of freedom

$$|\psi\rangle = \cos\theta\,|\uparrow\rangle_A|\downarrow\rangle_B + \sin\theta\,|\downarrow\rangle_A|\uparrow\rangle_B$$

Measuring bipartite entanglement in pure states

- Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j \, |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$, and $|\psi_j\rangle_A, |\psi_j\rangle_B$ orthonormal.

- one quantifier of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$

- if $c_1 = 1$, rest zero, $S = 0$ and $|\Psi\rangle$ is unentangled
- if all c_j equal, $S \sim \log \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$ – maximal entanglement

- equivalently, in terms of Alice's reduced density matrix:

$$\rho_A \equiv \mathrm{Tr}_B \, |\Psi\rangle\langle\Psi|$$

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A = S_B \qquad \text{von Neumann entropy}$$

- similar information is contained in the Rényi entropies

$$S_A{}^{(n)} = (1-n)^{-1} \log \mathrm{Tr}_A \, \rho_A{}^n$$

- $S_A = \lim_{n\to1} S_A{}^{(n)}$

- other measures of entanglement exist, but entropy has several nice properties: additivity, convexity, . . .
- it is monotonic under Local Operations and Classical Communication (LOCC)
- it gives the amount of classical information required to specify ρ_A (important for numerical computations)
- it gives a basis-independent way of identifying and characterising quantum phase transitions
- in a relativistic QFT the entanglement in the vacuum encodes all the data of the theory (spectrum, anomalous dimensions, . . .)

Entanglement entropy in QFT

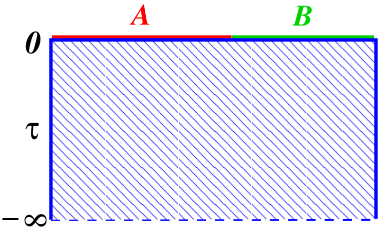
In this talk we consider the case when:

- the degrees of freedom are those of a local relativistic QFT in large region $\mathcal{R}$ in $\mathbb{R}^d$
- the whole system is in the vacuum state $|0\rangle$
- A is the set of degrees of freedom in some large (compact) subset of $\mathcal{R}$, so we can decompose the Hilbert space as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- in fact this makes sense only in a cut-off QFT (e.g. a lattice), and some of the results will in fact be cut-off dependent
- How does S_A depend on the size and geometry of A and the universal data of the QFT?

Rényi entropies from the path integral ($d = 1$)



- wave functional $\Psi(\{a\},\{b\}) \propto$ conditioned path integral in imaginary time from $\tau = -\infty$ to $\tau = 0$:

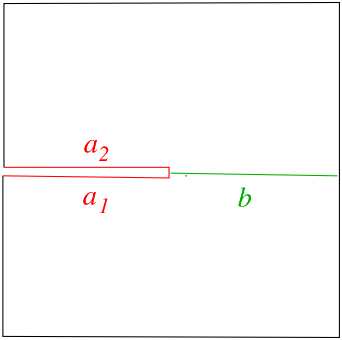
$$\Psi(\{a\},\{b\}) = Z_1^{-1/2} \int_{a(0)=a,b(0)=b} [da(\tau)][db(\tau)]\, e^{-(1/\hbar)S[\{a(\tau)\},\{b(\tau)\}]}$$

where $S = \int_{-\infty}^{0} L\big(a(\tau), b(\tau)\big)\, d\tau$

- similarly $\Psi^*(\{a\},\{b\})$ is given by the path integral from $\tau = 0$ to $+\infty$

$$\rho_A(a_1, a_2) = \int db\, \Psi(a_1, b)\Psi^*(a_2, b)$$

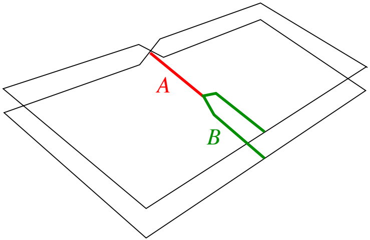
- this is given by the path integral over $\mathbb{R}^2$ cut open along $A \cap \{\tau = 0\}$, divided by Z_1:



Rényi entropies: example $n = 2$

$$\rho_A(a_1, a_2) = \int db\, \Psi(a_1, b)\Psi^*(a_2, b)$$

$$\mathrm{Tr}_A\, \rho_A^2 = \int da_1 da_2 db_1 db_2\, \Psi(a_1, b_1)\Psi^*(a_2, b_1)\Psi(a_2, b_2)\Psi^*(a_1, b_2)$$



$$\mathrm{Tr}_A\, {\rho_A}^2 = Z(\mathcal{R}_2)/Z_1^2$$

where $Z(\mathcal{R}_2)$ is the euclidean path integral (partition function) on an 2-sheeted conifold $\mathcal{R}_2$

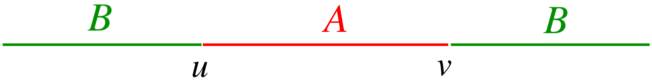
- in general

$$\mathrm{Tr}_A\, {\rho_A}^n = Z(\mathcal{R}_n)/Z_1^n$$

where the half-spaces are connected as

to form $\mathcal{R}_n$.

- conical singularity of opening angle $2\pi n$ at the boundary of A and B on $\tau = 0$



- if space is 1d and A is an interval (u, v) (and B is the complement) then $Z(\mathcal{R}_n)$ can be thought of as the insertion of twist operators into n copies of the CFT:

$$Z(\mathcal{R}_n)/Z_1^n = \langle \mathcal{T}_{-n}(u)\mathcal{T}_n(v)\rangle_{(CFT)^n}$$

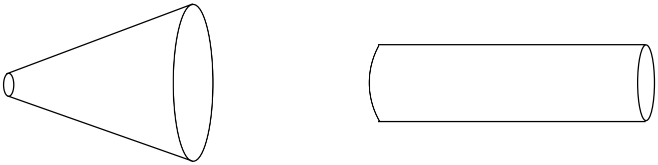
- these have similar properties to other local operators e.g.
- in a massless QFT (a CFT)

$$\langle \mathcal{T}_{-n}(u)\mathcal{T}_n(v)\rangle \sim |u-v|^{-2\Delta_n}$$

- in a massive QFT,

$$\langle \mathcal{T}_n\rangle \sim m^{\Delta_n} \quad \text{and} \quad \langle \mathcal{T}_{-n}(u)\mathcal{T}_n(v)\rangle - \langle \mathcal{T}_n\rangle^2 \sim \mathrm{e}^{-2m|u-v|}$$

- main result for $d = 1$: $\Delta_n = (c/12)(n - 1/n)$ where c is the central charge of the UV CFT



- consider a cone of radius R and opening angle $\alpha = 2\pi n$
- $w = \log z$ maps this into a cylinder of length $\log R$ and circumference α

$$\frac{Z_{\rm cone}(2\pi n)}{Z_{\rm cone}(2\pi)^n} = \frac{Z_{\rm cyl}(2\pi n)}{Z_{\rm cyl}(2\pi)^n} \sim \frac{e^{\pi c \log R/12\pi n}}{(e^{\pi c \log R/12\pi})^n} \sim R^{-\Delta_n}$$

- from this we see for example that for a single interval A of length ℓ [Holzhey, Larsen, Wilczek 1994]

$$S_A \sim -\left.\frac{\partial}{\partial n}\right|_{n=1} \ell^{-2\Delta_n} = (c/3)\log(\ell/\epsilon)$$

- note this is much less than the entanglement in a *typical* state which is $O(\ell)$
- many more universal results, eg finite-temperature cross-over between entanglement and thermodynamic entropy ($\beta = 1/k_B T$):

$$\begin{aligned} S_A &= (c/3)\log\left((\beta/\pi)\sinh(\pi\ell/\beta)\right) \\ &\sim (c/3)\log\ell \qquad \text{for } \ell \ll \beta \\ &\sim \pi c\ell/3\beta \qquad \text{for } \ell \gg \beta \end{aligned}$$

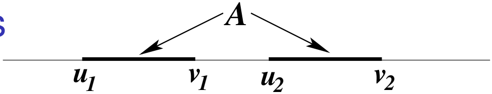
Massive QFT in 1+1 dimensions

- for 2 intervals $A = (-\infty, 0)$ and $B = (0, \infty)$

$$S_A \sim -(c/6) \log(m/\epsilon) \quad \text{as } m \to 0$$

- the entanglement diverges at a quantum phase transition and gives a basis-independent way of characterising the underlying CFT
- this is numerically the most accurate way of determining c for a given lattice model

Two intervals



$$Z(\mathcal{R}_n)/Z^n = \langle \mathcal{T}_{-n}(u_1)\mathcal{T}_n(v_1)\mathcal{T}_{-n}(u_2)\mathcal{T}_n(v_2)\rangle$$

- in general there is no simple result but for $|u_j - v_j| \ll |u_1 - u_2|$ we can use an operator product expansion

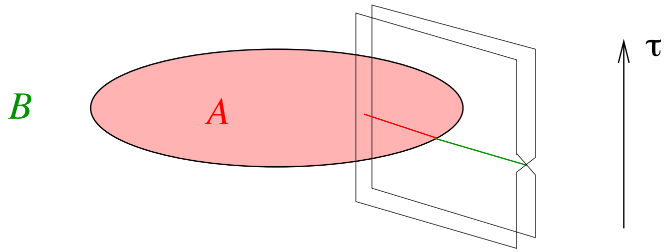
$$\mathcal{T}_{-n}(u)\mathcal{T}_n(v) = \sum_{\{k_j\}} C_{\{k_j\}}(u-v)\prod_{j=1}^{n}\Phi_{k_j}\left(\tfrac{1}{2}(u+v)_j\right)$$

  in terms of a complete set of local operators Φ_{k_j}
- this gives the Rényi entropies as an expansion

  $$\sum_{\{k_j\}} C^2_{\{k_j\}}\eta^{\sum_j \Delta_{k_j}}$$ where $\eta = ((u_1-v_1)(u_2-v_2))/((u_1-u_2)(v_1-v_2))$

- the $C_{\{k_j\}}$ encode all the data of the CFT

Higher dimensions $d > 1$



- the conifold $\mathcal{R}_n$ is now $\{2d$ conifold$\} \times \{$boundary $\partial A\}$

$$\log Z(\mathcal{R}_n) \sim \text{Vol}(\partial A) \cdot \epsilon^{-(d-1)}$$

- this is the 'area law' in 3+1 dimensions [Srednicki 1992]
- coefficient is non-universal

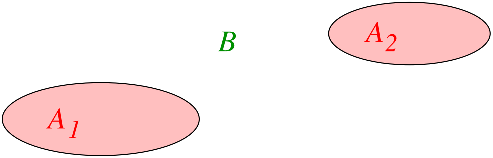
- for even $d+1$ there are interesting corrections to the area law

$$\text{Vol}(\partial A)\, m^2 \log(m\epsilon)\,, \quad a \log(R_A/\epsilon)$$

whose coefficients are related to curvature anomalies of the CFT and are universal

- e.g. a is the 'a-anomaly' which is supposed to decrease along RG flows between CFTs [Komargodski-Schwimmer]
- it would interesting to give an entanglement-based argument for this result [Casini-Huerta]

Mutual Information of multiple regions



- the non-universal ‘area’ terms cancel in

$$I^{(n)}(A_1, A_2) = S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1 \cup A_2}$$

- this mutual Rényi information is expected to be universal depending only on the geometry and the data of the CFT
- e.g. for a free scalar field in 3+1 dimensions [JC 2013]

$$I^{(n)}(A_1, A_2) \sim \frac{n^4 - 1}{15n^3(n-1)} \left(\frac{R_1 R_2}{r_{12}^2} \right)^2$$

Negativity

- however, mutual information does not correctly capture the quantum entanglement between A_1 and A_2, e.g. it also includes classical correlations at finite temperature
- more generally we want a way of quantifying entanglement in a mixed state $\rho_{A_1 \cup A_2}$
- one computable measure is negativity [Vidal, Werner 2002]
- let $\rho^{T_2}_{A_1 \cup A_2}$ be the *partial* transpose:

$$\rho^{T_2}_{A_1 \cup A_2}(a_1, a_2; a_1', a_2') = \rho_{A_1 \cup A_2}(a_1, a_2'; a_1', a_2)$$

- $\mathrm{Tr}\, \rho^{T_2}_{A_1 \cup A_2} = 1$, but it may now have negative eigenvalues λ_k

Log-negativity $$\mathcal{N} = \log \mathrm{Tr} \left| \rho^{T_2}_{A_1 \cup A_2} \right| = \log \sum_k |\lambda_k|$$

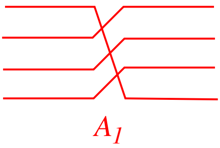
- if this is > 0 there are negative eigenvalues. This is an entanglement measure with nice properties, including being an LOCC monotone

Negativity in QFT

- ‘replica trick’

$$\mathrm{Tr}\,(\rho_{A_1\cup A_2}^{T_2})^n = \sum_k \lambda_k^n = \sum_k |\lambda_k|^n \quad \text{if } n \text{ is even}$$

- analytically continue to $n = 1$ to get $\sum_k |\lambda_k|$ (!!)
- we can compute $\mathrm{Tr}\,(\rho_{A_1\cup A_2}^{T_2})^n$ by connecting the half-spaces in the opposite order along A_2:

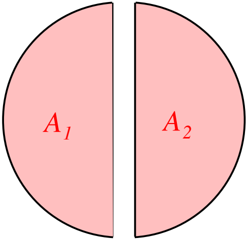


- for $\rho_{A_1\cup A_2}$ we need $\langle \mathcal{T}_{-n}(u_1)\mathcal{T}_n(v_1)\mathcal{T}_{-n}(u_2)\mathcal{T}_n(v_2)\rangle$
- for $\rho^{T_2}_{A_1\cup A_2}$ we need $\langle \mathcal{T}_{-n}(u_1)\mathcal{T}_n(v_1)\mathcal{T}_n(u_2)\mathcal{T}_{-n}(v_2)\rangle$

But

$$\begin{aligned} \mathcal{T}_n\cdot\mathcal{T}_n &\cong \mathcal{T}_n \quad (n\ \text{odd}) \qquad \to 1 \quad \text{for } n\to 1 \\ &\cong \mathcal{T}_{n/2}\otimes\mathcal{T}_{n/2} \quad (n\ \text{even}) \to \mathcal{T}_{1/2}\otimes\mathcal{T}_{1/2} \quad \text{for } n\to 1 \end{aligned}$$

so we get a non-trivial result if the intervals are close



- so for example for $d > 1$ for 2 large regions a finite distance apart

  $\mathcal{N}(A_1, A_2) \propto$ Area of common boundary between A_1 and A_2

- $\mathcal{N}$ appears to decay exponentially with separation of the regions, even in a CFT

Other Related Interesting Stuff

- topological phases in 2 (and higher) spatial dimensions - entanglement entropy distinguishes these in absence of local order parameter [Kitaev/Preskill and many others]
- 'entanglement spectrum' of the eigenvalues of $\log \rho_A$ [Haldane]
- Shannon entropy $-\mathrm{Tr}|\Psi|^2 \log |\Psi^2|$ seems to have interesting properties depute being basis-dependent
- holographic computation of entanglement using AdS/CFT [Ryu/Takayanagi and many others]
- time-dependence – in particular *quantum quenches* where the system is prepared in a state $|\psi\rangle$ which is not an eigenstate of hamiltonian: how do entanglement (and correlation functions) behave?