Problem Set 1

Although everyone in the class is welcome to try these problems, only the papers of students in Theoretical Physics (or any others taking the course for credit who have notified me) will be marked. They should be handed in by **Tuesday November 6 at 2pm**, either in the lecture that day or to Adam Nahum in Theoretical Physics. They will be reviewed at a Problems Class on **Wednesday November 14** from 3–5pm in the Dennis Sciama Lecture Theatre, DWB, which everyone in the class is welcome to attend. Any questions before the due date should be directed to me at `j.cardy1@physics.ox.ac.uk`

1. In the lecture we showed that the propagator for a free massive scalar euclidean field theory is

$$\Delta(x-x') = \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip\cdot(x-x')}}{p^2+m^2}\,.$$

   a) Show that as $|x-x'| \to \infty$ this behaves like $e^{-m|x-x'|}$, and also compute the prefactor in front of this.

   b) What is the corresponding result for Δ_F in Minkowski space? Distinguish between the answers for space-like and time-like separations.

2. A free scalar QFT with Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi) - \frac{1}{2}m^2\phi^2$ is in contact with a heat bath at inverse temperature $\beta = 1/kT$. By modifying the calculation we did in the lecture, calculate the equal-time correlation function $\langle\phi(\mathbf{x}_1,t)\phi(\mathbf{x}_2,t)\rangle$. [Note that because this is in fact independent of t you can equally well work in imaginary time, which is easier.] Show that as $T \to 0$ you get back the result we found in the lecture for the VEV. What happens at high temperatures $T \to \infty$?

3. Two coupled harmonic oscillators are described by the hamiltonian

$$\hat{H} = \tfrac{1}{2}(\hat{p}_1^2 + \omega^2\hat{q}_1^2) + \tfrac{1}{2}(\hat{p}_2^2 + \omega^2\hat{q}_2^2) + \lambda\hat{q}_1\hat{q}_2$$

The generating function in imaginary time is

$$Z(J_1, J_2) = \int [dq_1(\tau)][dq_2(\tau)] e^{-S_E[q_1,q_2] + \int (J_1(\tau)q_1(\tau) + J_2(\tau)q_2(\tau))d\tau}$$

where S_E is the action in imaginary time.

Evaluate $Z(J_1, J_2)/Z(0,0)$ by completing the square in the gaussian functional integral, and hence compute the correlation functions $\langle q_i(\tau_1)q_j(\tau_2)\rangle$. Show that when $\tau_1 = \tau_2$ these agree with the ground state expectation values $\langle 0|\hat{q}_i\hat{q}_j|0\rangle$ you find by solving the original problem by standard means.

4. What are the propagators (in momentum space) of the euclidean field theories with the following free actions? (You should be able to write these down more or less by inspection. There may be more than one propagator if there is more than one type of field.)

   a) $S = \frac{1}{2}\int[(\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z^2\phi)^2]dxdydz$ (note the ∂^2 in the last term)

   b) $S = \frac{1}{2}\int[(\partial\phi_1)^2 + (\partial\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2) + 2\lambda\phi_1\phi_2]d^dx$

5. What are the vertices corresponding to the following interaction terms in a scalar field theory? Draw the way they would appear at lowest order in a tree diagram, give the numerical factor, and label any lines as appropriate.

   a) $\lambda(\phi^{*3} + \phi^3) + \mu(\phi^{*2}\phi + \phi^*\phi^2)$ (ϕ is complex and the propagator is $\langle\phi\phi^*\rangle$).

   b) $\cos(\lambda\phi)$.

6. Consider a euclidean QFT with two real scalar fields ϕ and Φ and a lagrangian density

$$\tfrac{1}{2}((\partial\phi)^2 + m^2\phi^2) + \tfrac{1}{2}((\partial\Phi)^2 + M^2\Phi^2) + \tfrac{1}{2}\lambda\phi^2\Phi$$

   Write down the Feynman rules in momentum space for this theory (be careful to use a different sort of line for the propagators of different fields). Draw the tree and one loop diagrams that contribute to the correlation functions $\langle\phi\phi\rangle$, $\langle\Phi\Phi\rangle$, $\langle\Phi\rangle$ and $\langle\Phi\phi\phi\rangle$, and write down explicit expressions for the 1-loop diagrams, including the correct symmetry factors.