Introduction to Quantum Field Theory

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Michaelmas Term 2009

Abstract

These notes are intended to supplement the lecture course 'Introduction to Quantum Field Theory' and are not intended for wider distribution. Any errors or obvious omissions should be communicated to me at `j.cardy1@physics.ox.ac.uk`.

Contents

1 A Brief History of Quantum Field Theory

Quantum field theory (QFT) is a subject which has evolved considerably over the years and continues to do so. From its beginnings is elementary particle physics it has found applications in many other branches of science, in particular condensed matter physics but also as far afield as biology and economics. In this course we shall be adopting an approach (the *path integral*) which was not the original one, but became popular, even essential, with new advance in the 1970s. However, to set this in its context, it is useful to have some historical perspective on the development of the subject.

- 19th C. Maxwell's equations - a classical field theory for electromagnetism.
- 1900: Planck hypothesises the photon as the quantum of radiation.
- 1920s/30s: development of particle quantum mechanics: the same rules when applied to the Maxwell field predict photons. However relativistic particle quantum mechanics has problems (negative energy states.)
- 1930s/40s: realisation that relativity + quantum mechanics, in which particles can be created and destroyed, needs a many-particle descrip-

tion where the particles are the quanta of a quantised classical field theory, in analogy with photons.

- 1940s: formulation of the calculation rules for quantum electrodynamics (QED) – Feynman diagrams; the formulation of the path integral approach.
- 1950s: the understanding of how to deal with the divergences of Feynman diagrams through *renormalisation*; QFT methods begin to be applied to other many-body systems *eg* in condensed matter.
- 1960s: QFT languishes – how can it apply to weak + strong interactions?
- 1970s: renormalisation of non-Abelian gauge theories, the renormalisation group (RG) and asymptotic freedom; the formulation of the Standard Model
- 1970s: further development of path integral + RG methods: applications to critical behaviour.
- 1970s: non-perturbative methods, lattice gauge theory.
- 1980s: string theory + quantum gravity, conformal field theory (CFT); the realisation that all quantum field theories are only effective over some range of length and energy scales, and those used in particle physics are no more fundamental than in condensed matter.
- 1990s/2000s: holography and strong coupling results for gauge field theories; many applications of CFT in condensed matter physics.

Where does this course fit in?

In 16 lectures, we cannot go very far, or treat the subject in much depth. In addition this course is aimed at a wide range of students, from experimental particle physicists, through high energy theorists, to condensed matter physicists (with maybe a few theoretical chemists and mathematicians thrown in). Therefore all I can hope to do is to give you some of the basic ideas, illustrated in their most simple contexts. The hope is to take you all from the Feynman path integral, through a solid grounding in Feynman diagrams, to renormalisation and the RG. From there hopefully you will have enough background to understand Feynman diagrams and their uses in particle physics, and have the basis for understanding gauge theories as well as applications of field theory and RG methods in condensed matter physics.

2 The Feynman path integral in particle quantum mechanics

In this lecture we will recall the Feynman path integral for a system with a single degree of freedom, in preparation for the field theory case of many degrees of freedom.

Consider a non-relativistic particle of unit mass moving in one dimension. The coordinate operator is $\hat{q}$, and the momentum operator is $\hat{p}$. (I'll be careful to distinguish operators and c-numbers.) Of course $[\hat{q},\hat{p}] = i\hbar$. We denote the eigenstates of $\hat{q}$ by $|q'\rangle$, thus $\hat{q}|q'\rangle = q'|q'\rangle$, and $\langle q'|q''\rangle = \delta(q'-q'')$.

Suppose the hamiltonian has the form $\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{q})$ (we can consider more general forms – see later.) The *classical* action corresponding to this is

$$S[q] = \int_{t_i}^{t_f} \left[\tfrac{1}{2}\dot{q}^2 - V(q(t))\right] dt$$

where $q(t)$ is a possible classical trajectory, or path. According to Hamilton's principle, the actual classical path is the one which extremises S – this gives Lagrange's equations.

The quantum amplitude for the particle to be at q_f at time t_f given that it was at q_i at time t_i is

$$M = \langle q_f|e^{-i\hat{H}(t_f-t_i)/\hbar}|q_i\rangle\,.$$

According to Feynman, this amplitude is equivalently given by the path integral

$$I = \int [dq]\, e^{iS[q]/\hbar}$$

which is a integral over all functions (or paths) $q(t)$ which satisfy $q(t_i) = q_i$, $q(t_f) = q_f$. Obviously this needs to be better defined, but we will try to make sense of it as we go along.

In order to understand why this might be true, first split the interval (t_i, t_f) into smaller pieces

$$(t_f, t_{n-1}, \ldots, t_{j+1}, t_j, \ldots, t_1, t_i)$$

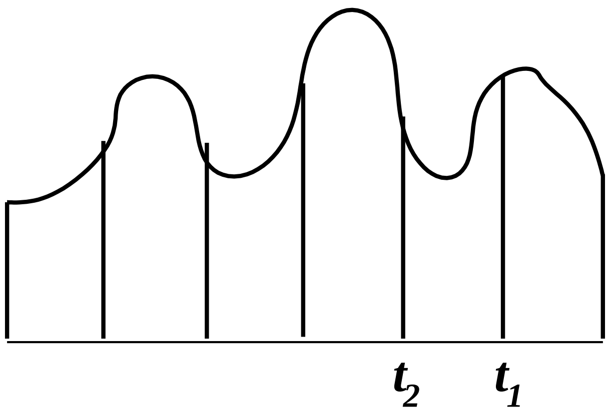

Figure 1: We can imagine doing the path integral by first fixing the values of $q(t)$ at times $(t_1, t_2, \ldots)$.

with $t_{j+1} - t_j = \Delta t$. Our matrix element can then be written

$$M = \langle q_f | \overbrace{e^{-i\hat{H}\Delta t/\hbar} \ldots e^{-i\hat{H}\Delta t/\hbar}}^{N \text{ factors}} | q_i \rangle$$

(Note that we could equally well have considered a time-dependent hamiltonian, in which case each factor would be different.) Now insert a complete set of eigenstates of $\hat{q}$ between each factor, *eg* at time-slice t_j insert

$$\int_{-\infty}^{\infty} dq(t_j) |q(t_j)\rangle\langle q(t_j)|$$

so that

$$M = \prod_j \int dq(t_j) \langle q(t_{j+1})| e^{-i\hat{H}\Delta t/\hbar} |q(t_j)\rangle$$

On the other hand, we can think of doing the path integral $\int [dq]$ by first fixing the values $\{q(t_j)\}$ at times $\{t_j\}$ (see Fig. 1) and doing the integrals over the intermediate points on the path, and then doing the integral over the $\{q(t_j)\}$. Thus

$$I = \prod_j \int dq(t_j)\, e^{(i/\hbar) \int_{t_j}^{t_{j+1}} \left(\frac{1}{2}\dot{q}^2 - V(q(t))\right) dt}$$

Thus we can prove that $M = I$ in general if we can show that

$$\langle q(t_{j+1})| e^{-i\hat{H}\Delta t/\hbar} |q(t_j)\rangle = e^{(i/\hbar) \int_{t_j}^{t_{j+1}} \left(\frac{1}{2}\dot{q}^2 - V(q(t))\right) dt}$$

for an arbitrarily short time interval Δt. First consider the case when $V = 0$. The path integral is

$$\int [dq] e^{(i/2\hbar) \int_{t_j}^{t_{j+1}} \dot{q}^2 dt}$$

Let $q(t) = q_c(t) + \delta q(t)$ where $q_c(t)$ interpolates linearly between $q(t_j)$ and $q(t_{j+1})$, that is

$$q_c(t) = q(t_j) + (\Delta t)^{-1}(t - t_j)(q(t_{j+1}) - q(t_j))$$

and $\delta q(t_{j+1}) = \delta q(t_j) = 0$. Then

$$\int_{t_j}^{t_{j+1}} \dot{q}^2 dt = (\Delta t)\left(\frac{q(t_{j+1}) - q(t_j)}{\Delta t}\right)^2 + \int (\delta \dot{q})^2 dt$$

and

$$\int [dq] e^{(i/2\hbar)\int_{t_j}^{t_{j+1}} \dot{q}^2 dt} = e^{i\left(q(t_{j+1}) - q(t_j)\right)^2/2\hbar\Delta t} \int [d(\delta q] e^{(i/2\hbar)\int(\delta\dot{q})^2 dt}$$

The second factor depends on Δt but not $q(t_{j+1})$ or $q(t_j)$, and can be absorbed into the definition, or normalisation, of the functional integral. The first factor we recognise as the spreading of a wave packet initially localised at $q(t_j)$ over the time interval Δ. This is given by usual quantum mechanics as

$$\langle q(t_{j+1})| e^{-i\hat{p}^2\Delta t/2\hbar} | q(t_j)\rangle$$

(and this can be checked explicitly using the Schrödinger equation.)

Now we argue, for $V \neq 0$, that if Δt is small the spreading of the wave packet is small, and therefore we can approximate $V(q)$ by (say) $V(q(t_j))$. Thus, as $\Delta t \to 0$,

$$\int [dq] e^{(i/\hbar)\int_{t_j}^{t_{j+1}} \left(\frac{1}{2}\dot{q}^2 - V(q(t))\right) dt} \sim \langle q(t_{j+1})| e^{-i(\Delta t/\hbar)\left(\frac{1}{2}\hat{q}^2 + V(\hat{q})\right)} | q(t_j)\rangle$$

Putting all the pieces together, an integrating over the $\{q(t_j)\}$, we obtain the result we want.

As well as being very useful for all sorts of computations, the path integral also provides an intuitive way of thinking about classical mechanics as a limit of quantum mechanics. As $\hbar \to 0$ in the path integral $\int [dq] e^{iS[q]/\hbar}$, the important paths are those corresponding to *stationary phase*, where $\delta S[q]/\delta q = 0$. Other paths giving rapidly oscillating contributions and therefore are suppressed. This is just Hamilton's principle. In the *semi-classical* limit, the important paths will be those close to the classical one. *Periodic* classical orbits will carry a complex phase which will in general average to zero over many orbits. However if the action of a single orbit is $2\pi\hbar\times$ integer, the phase factor is unity and therefore such orbits will dominate the path integral. This is the Bohr-Sommerfeld quantisation condition.

The path integral is not restricted to hamiltonians of the above form, but is more general. An important case is when $\hat{H}(\hat{a},\hat{a}^\dagger)$ is expressed in terms of annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ satisfying $[\hat{a},\hat{a}^\dagger]=1$. In this case, the path integral is obtained by replacing these by complex-valued functions $a(t)$ and $a^*(t)$:

$$\int [da][da^*]e^{(i/\hbar)\int\left(i\hbar a^*\partial_t a - H(a,a^*)\right)dt}$$

This is called a coherent state path integral. Similar versions exist for hamiltonians depending on quantum spins.

2.1 Imaginary time path integrals and statistical mechanics

Sometimes it is useful to consider matrix elements of the form

$$M = \langle q_f|e^{-\hat{H}(\tau_f-\tau_i)/\hbar}|q_i\rangle\,, \tag{1}$$

that is, without the i. An analogous argument to the above shows that this is given by the path integral

$$\int [dq]e^{-S_E[q]/\hbar} \tag{2}$$

where

$$S_E[q] = \int_{\tau_i}^{\tau_f} (\tfrac{1}{2}\dot{q}^2 + V(q(\tau)))d\tau$$

This is called the 'imaginary time' path integral: if we formally let $t=-i\tau$ in the previous result, we get this answer. For reasons that will become apparent in the field theory generalisation, S_E is usually referred to as the euclidean action. Note that the relative sign of the kinetic and potential terms changes between S and S_E.

One application of this idea is to quantum statistical mechanics. The canonical partition function in general is

$$Z = \operatorname{Tr} e^{-\beta\hat{H}}$$

where $\beta = 1/k_BT$. For the model under consideration the trace can be written

$$Z = \int dq_i \langle q_i|e^{-\beta\hat{H}}|q_i\rangle$$

where the matrix element is of the form (1) with $\tau_f - \tau_i = \beta\hbar$. Thus Z is also given by the imaginary time path integral (2) over *periodic* paths satisfying $q(\tau_i + \beta\hbar) = q(\tau_i)$.

Another application is to the computation of the ground state energy E_0. If we insert a complete set of eigenstates of $\hat{H}$ into (1) in the limit $\tau_f - \tau_i \equiv T \to \infty$, the leading term has the form $\sim e^{-E_0 T}$. On the other hand, in (2) this is given by paths $q(\tau)$ which minimise $S_E[q]$. Typically they must satisfy $\dot{q}(\tau) \to 0$ as $\tau \to \pm\infty$. In most cases these have $\dot{q} = 0$ throughout, but other cases are more interesting. In particular this leads to an understanding of quantum-mechanical tunnelling (see Problems.)

The imaginary time path integral (2) may also be though of as a partition function in *classical* statistical mechanics. Suppose that we treat τ as a spatial coordinate, and $q(\tau)$ as the transverse displacement of a stretched elastic string tethered at the points τ_i and τ_f. In addition a force, described by an external potential $V(q)$, acts on the string. The euclidean action

$$S_E[q] = \int (\tfrac{1}{2}m(dq/d\tau)^2 + V(q(\tau)))d\tau$$

(where we have restored the particle mass m in the original problem) can now be thought of as the *potential* energy of the string, the first term representing the bending energy where m is the string tension. The partition function of the string in classical statistical mechanics is

$$Z = \int [dq][d\dot{q}]e^{-\left(\int \frac{1}{2}\rho\dot{q}^2 d\tau + S_E[q]\right)/k_B T}$$

where $\dot{q}$ now means the derivative wrt real time and $\int \frac{1}{2}\rho\dot{q}^2 d\tau$ is the kinetic energy, with ρ being the string's mass per unit length. The integral over $\dot{q}$ just gives a constant, so comparing with (2) we see that the imaginary time path integral actually corresponds to a classical partition function at temperature $k_B T = \hbar$. This is the simplest example of one of the most powerful ideas of theoretical physics:

$\Rightarrow$Quantum mechanics (in imaginary time) $\equiv$ classical statistical mechanics in one higher spatial dimension$\Leftarrow$

3 Path integrals in field theory

A field theory is a system whose degrees of freedom are distributed throughout space. Since the continuous version of this is a little difficult to grasp initially, consider a discrete regular lattice in D-dimensional space whose sites are labelled by $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)$. At each site there is a degree of freedom. Instead of $\hat{q}$ and $\hat{p}$ we use $\hat{\phi}$ and $\hat{\pi}$. Thus

$$\begin{aligned} \hat{q} &\rightarrow (\hat{\phi}(\mathbf{x}_1), \hat{\phi}(\mathbf{x}_2), \ldots) \\ \hat{p} &\rightarrow (\hat{\pi}(\mathbf{x}_1), \hat{\pi}(\mathbf{x}_2), \ldots) \end{aligned}$$

satisfying the canonical commutation relations

$$[\hat{\phi}(\mathbf{x}_j), \hat{\pi}(\mathbf{x}_{j'})] = i\hbar\delta_{jj'}$$

(Note that, in field theory, it is common to denote coordinates by simple italics even though these are D-component vectors.) The simplest form of the hamiltonian, generalising our single degree of freedom example, is

$$\hat{H} = \sum_j \hat{h}(\hat{\pi}(\mathbf{x}_j), \hat{\phi}(\mathbf{x}_j)) + \tfrac{1}{2}J \sum_{(jj')} (\hat{\phi}(\mathbf{x}_j) - \hat{\phi}(\mathbf{x}_{j'}))^2$$

where the last term couples the degrees of freedom on neighbouring sites. We can take $\hat{h}$ to have the same form as before,

$$\hat{h}(\hat{\pi}(\mathbf{x}_j), \hat{\phi}(\mathbf{x}_j)) = \tfrac{1}{2}\hat{\pi}(\mathbf{x}_j)^2 + V(\hat{\phi}(\mathbf{x}_j))$$

In the path integral version the operators $\hat{\phi}(\mathbf{x}_j)$ are replaced by c-number variables $\phi(\mathbf{x}_j, t)$:

$$\int \prod_j [d\phi(\mathbf{x}_j, t)]\, e^{(i/\hbar)S[\{\phi(\mathbf{x}_j,t)\}]}$$

where

$$S = \int \left(\sum_j (\tfrac{1}{2}\dot{\phi}(\mathbf{x}_j, t)^2 - V(\phi(\mathbf{x}_j, t))) - \tfrac{1}{2}J \sum_{(jj')} (\phi(\mathbf{x}_j, t) - \hat{\phi}(\mathbf{x}_{j'}, t))^2 \right) dt$$

This is the action for a lattice field theory.

However we are interested in the continuum limit, as the lattice spacing $a \rightarrow 0$. The *naive* continuum limit is obtained by replacing sums over lattice sites by integrals:

$$\sum_j \rightarrow \int \frac{d^D x}{a^D}$$

and making a gradient (Taylor) expansion of finite differences:

$$\sum_{(jj')} (\phi(\mathbf{x}_j,) - \hat{\phi}(\mathbf{x}_{j'}, t))^2 \rightarrow \int \frac{d^D x}{a^D}\, a^2(\nabla\phi(\mathbf{x}, t))^2$$

After rescaling $\phi \to J^{-1/2}a^{(D-2)/2}\phi$ (and also t), the action becomes

$$S = \int dt d^D x \left(\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi)\right)$$

This is the action for a classical field theory. The quantum theory is given by the path integral over fields $\phi(\mathbf{x}, t)$

$$\int [d\phi(\mathbf{x}, t)]\, e^{iS[\phi]/\hbar}$$

However, this begs the question of whether this has a meaningful limit as $a \to 0$. The naive answer is no, and making sense of this limit requires the understanding of renormalisation.

3.1 Field theory action functionals

The example that we discussed above has several nice properties:

- it is *local* : this means that S can be written as $\int \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) dt d^D x$ where the lagrangian density depends on the local value of the field and its derivatives. Moreover (more technically) it depends on derivatives only up to second order. It can be shown that higher order derivatives in t lead to violations of causality.

- it is relativistically invariant (with $c = 1$): in 4-vector (or D+1-vector) notation $\mathcal{L}$ can be written

$$\mathcal{L} = \tfrac{1}{2}(\partial_0\phi)^2 - \tfrac{1}{2}\sum_i(\partial_i\phi)^2 - V(\phi) = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)$$

  so that if ϕ transforms as a Lorentz scalar, $\mathcal{L}$ is Lorentz invariant. This is of course a requirement for a field theory describing relativistic particles. Another example is

$$\mathcal{L} = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

  where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is a Lorentz vector. This is the lagrangian for the electromagnetic field. However in condensed matter physics applications, relativistic invariance is not necessary (although it sometimes emerges anyway, with c replaced by the Fermi velocity or the speed of sound.) Note also that the imaginary time version of the action for our scalar field theory is

$$S_E = \int \left(\tfrac{1}{2}\sum_{i=1}^{d}(\partial_i\phi)^2 + V(\phi)\right) d^d x$$

where $d = D + 1$ and $d^d x = d^D x d\tau$. That is, τ plays the same role as a spatial coordinate and the theory is invariant under rotations in d-dimensional euclidean space. For this reason the imaginary time versions are called euclidean quantum field theories.

- $\mathcal{L}$ should be invariant under any internal symmetries of the theory. If this is $\phi \to -\phi$, for example, then V should be an even function. In the case of electromagnetism, the symmetry is local gauge invariance.

3.2 The generating functional

One difference between single particle quantum mechanics and quantum field theory is that we are not usually interested in transition amplitudes between eigenstates $|\phi(\mathbf{x})\rangle$ of the field itself, as the field itself is not physically measurable. In fact, since we usually consider the limit of infinite space, on relativistic grounds we should also consider infinite times. Thus the only meaningful path integral would seem to be

$$\int [d\phi] e^{(i/\hbar)\int_{-\infty}^{\infty} dt \int \mathcal{L} d^D x} \tag{3}$$

which is just a number. In fact, if we consider the euclidean version of this,

$$\int [d\phi] e^{-(1/\hbar)\int_{-\infty}^{\infty} d\tau \int \mathcal{L} d^D x} \tag{4}$$

and relate this to a matrix element between eigenstates $|n\rangle$ of $\hat{H}$, we get

$$\lim_{\tau_f - \tau_i \to \infty} \sum_n e^{-E_n(\tau_f - \tau_i)} \langle n|n\rangle \sim e^{-E_0(\tau_f - \tau_i)} \langle 0|0\rangle$$

Thus we see that (4) (and, by careful definition through analytic continuation, see later, (3)) just tells about the vacuum $\to$ vacuum amplitude, and is thus not very interesting (at least in flat space.)

In order to get any interesting physics we have to 'tickle' the vacuum, by adding sources which can make things happen. The simplest and most useful way of doing this is to add a source coupling locally to the field itself, that is change the action to

$$S \to S + \int J(x)\phi(x) d^d x$$

The vacuum amplitude is now a functional of this source function $J(x)$:

$$Z[J] = \int [d\phi] e^{iS + i\int J(x)\phi(x) d^d x}$$

We are now using $x = (\mathbf{x}, t)$ to represent a point in Minkowski space, and we have started using units where $\hbar = 1$, both standard conventions in QFT.

Since the i makes this rather ill-defined, we shall, for the time being, develop the theory in the euclidean version

$$Z[J] = \int [d\phi] e^{-S + \int J(x)\phi(x) d^d x}$$

Interesting physical quantities are found by taking functional derivatives of $Z[J]$ with respect to J. For example

$$\frac{1}{Z[0]} \frac{\delta Z[J]}{\delta\phi(x_1)}\bigg|_{J=0} = \frac{1}{Z[0]} \int [d\phi] \phi(x_1)\, e^{-S[\phi]}$$

By analogy with statistical mechanics in d dimensions, this can be thought of as an expectation value

$$\langle \phi(x_1) \rangle$$

Similarly

$$\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{J=0} = \frac{1}{Z[0]} \int [d\phi] \phi(x_1)\phi(x_2)\, e^{-S[\phi]} = \langle \phi(x_1)\phi(x_2) \rangle\,,$$

a correlation function.

But what do these mean in the operator formulation? To see this imagine inserting a complete set of eigenstates. Then as $\tau_i \to \infty$ and $\tau_f \to +\infty$,

$$\int [d\phi] \phi(x_1)\, e^{-S[\phi]} \sim e^{-E_0(\tau_f - \tau_i)} \langle 0 | \hat{\phi}(x_1) | 0 \rangle$$

and the first factor gets cancelled by $Z[0]$. Similarly the two-point function is

$$\langle \phi(x_1)\phi(x_2) \rangle = \langle 0 | \hat{\phi}(\mathbf{x}_1) e^{-(\hat{H} - E_0)(\tau_1 - \tau_2)} \hat{\phi}(\mathbf{x}_2) | 0 \rangle$$

where we have emphasised that $\hat{\phi}$, in the Schrödinger picture, depends on the spatial coordinates $\mathbf{x}$ but not τ. However if we go to the Heisenberg picture and define

$$\hat{\phi}(x) = e^{-(\hat{H} - E_0)\tau}\, \hat{\phi}(\mathbf{x})\, e^{(\hat{H} - E_0)\tau}$$

the rhs becomes

$$\langle 0 | \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle\,.$$

However this is correct only if $\tau_1 > \tau_2$. If the inequality were reversed we would have had to write the factors in the reverse order. Thus we conclude that

$$\frac{1}{Z[0]}\frac{\delta^2 Z[J]}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{J=0} = \langle\phi(x_1)\phi(x_2)\rangle = \langle 0|\mathbf{T}\left[\hat{\phi}(x_1)\hat{\phi}(x_2)\right]|0\rangle$$

where $\mathbf{T}$ arranges the operators in order of decreasing τ.

⇒Functional derivatives of $Z[J]$ given vacuum expectation values of time-ordered products of field operators⇐

This result continues to hold when we go back to real time t. Fortunately it is precisely these vacuum expectation values of time-ordered products which arise when we do scattering theory.

In field theory, the correlation functions are also called Green functions (as we'll see, for a free field theory they are Green functions of differential operators), or simply the N-point functions

$$G^{(N)}(x_1,\ldots,x_N) = \langle\phi(x_1)\ldots\phi(x_N)\rangle = \frac{1}{Z[0]}\,\frac{\delta^N Z[J]}{\delta J(x_1)\ldots\delta J(x_N)}\bigg|_{J=0}$$

Equivalently

$$\frac{Z[J]}{Z[0]} = \sum_{N=0}^{\infty}\frac{1}{N!}\int d^dx_1\ldots\int d^dx_N\,G^{(N)}(x_1,\ldots,x_N)J(x_1)\ldots J(x_N)$$

$Z[J]$ is called the *generating function* for the N-point functions.

It is also useful to define

$$W[J] \equiv \log Z[J]\,,$$

which is analogous to the free energy in statistical mechanics. We expect, by analogy, that $W[0]$ is proportional to the total space-time volume VT, and that, if the sources J are localised to a finite region of space-time, that $W[J] - W[0]$ is finite in the limit $VT \to \infty$. Thus functional derivatives of W wrt J should also be finite. These give what are called the *connected* correlation functions $\langle\phi(x_1)\ldots\phi(x_N)\rangle_c$ or $G^{(N)}(x_1,\ldots,x_N)_c$. The reason for this will become apparent when we write them in terms of Feynman diagrams. For example

$$\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \langle\phi(x_1)\phi(x_2)\rangle_c = \langle\phi(x_1)\phi(x_2)\rangle - \langle\phi(x_1)\rangle\langle\phi(x_2)\rangle$$

$W[J]$ is the generating function for the connected N-point functions.

3.3 The propagator in free field theory

The only path integrals we can actually do (except in certain esoteric theories with supersymmetry) are gaussian, that is when the action S is at most quadratic in the field ϕ. However this is an important case, corresponding to a free field theory. As usual, we consider the euclidean case first.

$$Z_0[J] = \int [d\phi]\, e^{-\int \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right] d^dx + \int J(x)\phi(x) d^dx}$$

(So far m is just a parameter, but it will turn out that in Minkowski space this theory describes free relativistic particles of mass m.) The first term can be integrated by parts to give $\frac{1}{2}\phi(-\partial^2)\phi$.

Define Fourier transforms:

$$\begin{aligned} \tilde{\phi}(p) &= \int d^dx\, e^{-ip\cdot x}\, \phi(x) \\ \phi(x) &= \int \frac{d^dp}{(2\pi)^d}\, e^{ip\cdot x}\, \tilde{\phi}(p) \end{aligned}$$

and similarly for $\tilde{J}(p)$ and $J(x)$. (Note that in field theory it is conventional to put the factors of 2π as above.)

The negative of the expression in the exponential is then

$$\int \frac{d^dp}{(2\pi)^d} \left[\tfrac{1}{2}\tilde{\phi}(p)(p^2+m^2)\tilde{\phi}(-p) - \tilde{J}(p)\tilde{\phi}(-p)\right]$$

Completing the square on the expression in square brackets:

$$\begin{aligned} &\frac{1}{2}\left[\tilde{\phi}(p) - \frac{1}{p^2+m^2}\tilde{J}(p)\right](p^2+m^2)\left[\tilde{\phi}(-p) - \frac{1}{p^2+m^2}\tilde{J}(-p)\right] \\ &-\frac{1}{2}\tilde{J}(p)\frac{1}{p^2+m^2}\tilde{J}(-p) \end{aligned}$$

Now the functional integral $\int[d\phi(x)]$ can equally well be carried out over $\int[d\tilde{\phi}(p)]$. Shifting the integration variable $\tilde{\phi}(p) = \tilde{\phi}(p)' + (p^2+m^2)^{-1}\tilde{J}(p)$ gives

$$Z_0[J] = \int [d\tilde{\phi}']\, e^{-\frac{1}{2}\int (d^dp/(2\pi)^d)\tilde{\phi}(p)'(p^2+m^2)\tilde{\phi}(-p)' + \frac{1}{2}\int (d^dp/(2\pi)^d)\tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

The first term in the exponential gives a factor independent of J, so

$$Z_0[J] = Z_0[0]\, e^{\frac{1}{2}\int (d^dp/(2\pi)^d)\tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

Going back to coordinate space

$$Z_0[J] = Z_0[0]\, e^{\frac{1}{2}\int d^dx' \int d^dx'' J(x')\Delta(x'-x'')J(x'')}$$

where

$$\Delta(x'-x'') \equiv \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip\cdot(x'-x'')}}{p^2+m^2}$$

With this result in hand we can now compute correlation functions in the free theory, *eg*

$$\langle\phi(x_1)\rangle_0 = \tfrac{1}{2}\int d^dx''\Delta(x_1-x'')J(x'') + \tfrac{1}{2}\int d^dx'\Delta(x'-x_1)J(x')\Big|_{J=0} = 0$$

$$\langle\phi(x_1)\phi(x_2)\rangle_0 = \tfrac{1}{2}\Delta(x_1-x_2) + \tfrac{1}{2}\Delta(x_2-x_1) = \Delta(x_1-x_2)$$

$\Delta(x_1-x_2)$ is thus the 2-point function $G_0^{(2)}(x_1,x_2)$ in the free theory.

$\langle\phi(x_1)\ldots\phi(x_N)\rangle_0 = 0$ if N is odd, but, for example

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_3)\rangle_0 \;=\; & \Delta(x_1-x_2)\Delta(x_3-x_4) \\ & +\Delta(x_1-x_3)\Delta(x_2-x_4) \\ & +\Delta(x_1-x_4)\Delta(x_2-x_3)\end{aligned}$$

In general, for N even,

$$\langle\phi(x_1)\ldots\phi(x_N)\rangle_0 = \sum \Delta(x_{j_1}-x_{j'_1})\ldots\Delta(x_{j_{N/2}}-x_{j'_{N/2}})$$

where the sum is over all distinct ways of grouping the set $\{1,2,\ldots,N\}$ into pairs. This result, which in fact holds for any gaussian integral, is the path integral version of *Wick's theorem*. It tells us that in the free theory, every correlation function can be expressed in terms of $G_0^{(2)}$. Another way of stating it is to observe that the generating function for connected correlation functions $W[J]$ is quadratic in J. Thus all connected N-point functions vanish for $N > 2$.

At this stage we can begin to introduce a graphical notation which will become one of the building blocks for Feynman diagrams. We denote $\Delta(x_1-x_2)$ by an (unoriented) line connecting the points x_1 and x_2, as shown in Fig. 2. (it doesn't matter exactly where we put the points, only the topology is important.) Then Wick's theorem for $N = 4$ can be expressed by connecting up the points (x_1, x_2, x_3, x_4) by lines in all possible ways, such that exactly one line ends at each point. See Fig.3.

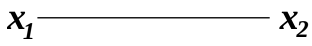

Figure 2: Graphical representation of the propagator $\Delta(x_1 - x_2)$.

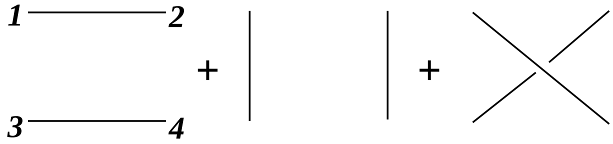

Figure 3: Wick contractions for the 4-point function. Each line represents a factor Δ.

3.3.1 Minkowski space

In real time, the path integral is less well-defined, because the integrand is oscillating rather than exponentially damped at large values of ϕ:

$$Z_0[J] = \int [d\phi]\, e^{i\int \left(\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right)dtd^D\mathbf{x} + i\int J\phi dtd^D\mathbf{x}}$$

where $(\partial_\mu\phi)(\partial^\mu\phi) = (\partial_t\phi)^2 - (\nabla\phi)^2$.

One way to make this better defined is to give the parameter m^2 a small negative imaginary part

$$m^2 \to m^2 - i\epsilon$$

Now that the integral is absolutely convergent we can rotate the contour in the t-integration by letting $t = -i\tau$, whereby

$$Z_0[J] = \int [d\phi]\, e^{-\int \left(\frac{1}{2}((\partial_\tau\phi)^2 + (\nabla\phi)^2) + \frac{1}{2}(m^2 - i\epsilon)\phi^2\right)d^dx + \int J\phi d^dx}$$

This is the generating function in euclidean space. So we can get all the results in Minkowski space by substituting $\tau = it$ in their euclidean versions. This technique is called *Wick rotation.* Note that when we do this,

$$p \cdot x = p_0\tau + \mathbf{p}\cdot\mathbf{x} \to ip_0 t + \mathbf{p}\cdot\mathbf{x}$$

so that we have to let $p_0 \to ip_0$ and then

$$p \cdot x\,(\text{euclidean}) \to -p_\mu x^\mu\,(\text{Minkowski})$$

Thus

$$\begin{aligned} Z_0[J] &= Z_0[0]\, e^{-(i/2)\int (d^dp/(2\pi)^d)\widetilde{J}(p)(p^2 - m^2 + i\epsilon)^{-1}\widetilde{J}(-p)} \\ &= e^{-(i/2)\int d^dx' d^dx'' J(x')\Delta_F(x' - x'')J(x'')} \end{aligned}$$

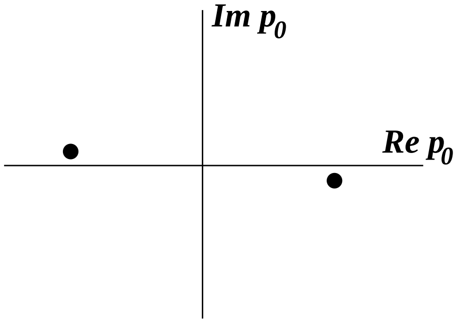

Figure 4: Poles of the Feynman propagator in the complex p_0-plane.

where

$$\Delta_F(x_1 - x_2) = \int \frac{d^dp}{(2\pi)^d} \frac{e^{-ip_\mu(x_1^\mu - x_2^\mu)}}{p^2 - m^2 + i\epsilon} \tag{5}$$

Δ_F is called the *Feynman propagator.* We shall discuss its physical interpretation below. If we recall that

$$G^{(2)}(x_1, x_2) = \frac{\delta^2 Z[J]}{\delta(iJ(x_1))\delta(iJ(x_2))}$$

(note the factors of i), we see that

$$G_0^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2)$$

Let us examine the p_0 integration in (5):

$$\int_{-\infty}^{\infty} \frac{e^{-ip_0(t_1-t_2)}}{p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon} \frac{dp_0}{2\pi}$$

The integrand has poles at $p_0 = \pm\sqrt{\mathbf{p}^2 + m^2 - i\epsilon}$ (see Fig. 4). Suppose that $t_1 > t_2$. Then we can close the p_0 contour in the lower half plane, picking up the pole with the positive sign of Re p_0. This gives

$$G_0^{(2)}(x_1, x_2) = \int \frac{d^Dp}{(2\pi)^D} \frac{e^{-i\sqrt{\mathbf{p}^2+m^2}(t_1-t_2)+i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{2\sqrt{\mathbf{p}^2 + m^2}}$$

If $t_2 > t_1$ we pick up the other pole and get the same result with t_1 and t_2 interchanged. Now recall that in the operator formulation is a vacuum expectation value of a time-ordered product:

$$G_0^{(2)}(x_1, x_2) = \langle 0|\mathbf{T}[\hat{\phi}(x_1)\hat{\phi}(x_2)]|0\rangle$$

Thus if we define

$$\tilde{\hat{\phi}}(\mathbf{p}, t) = \int d^Dx\, e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{\phi}(\mathbf{x}, t)$$

we see that, for $t_1 > t_2$,

$$\langle 0|\tilde{\hat{\phi}}(-\mathbf{p_1}, t_1)\tilde{\hat{\phi}}(\mathbf{p_2}, t_2)|0\rangle = (2\pi)^D \delta(\mathbf{p}_1 - \mathbf{p}_2)\frac{e^{-i\sqrt{\mathbf{p}_2^2+m^2}(t_1-t_2)}}{2\sqrt{\mathbf{p}_2^2+m^2}}$$

The interpretation of this is that $\tilde{\hat{\phi}}(\mathbf{p_2}, t_2)$ creates a particle of momentum $\mathbf{p}_2$ and energy $\sqrt{\mathbf{p}_2^2 + m^2}$ at time t_2, and $\tilde{\hat{\phi}}(-\mathbf{p_2}, t_1)$ destroys this particle. The rhs is the quantum amplitude for the particle to propagate from x_2 to x_1, and is therefore called the *propagator*.

4 Interacting field theories

4.1 Feynman diagrams

4.2 Evaluation of Feynman diagrams

5 Renormalisation

5.1 Analysis of divergences

5.2 Mass, field, and coupling constant renormalisation

6 Renormalisation Group

6.1 Callan-Symanzik equation

6.2 Renormalisation group flows

6.3 One-loop computation in $\lambda\Phi^4$ theory

6.4 Application to critical behaviour in statistical mechanics

6.5 Large N

7 From Feynman diagrams to Cross-sections

7.1 The S-matrix: analyticity and unitarity

8 Path integrals for fermions