QFT 1. Problem Set 1. Solutions MT 2009.

1 a). According to what we did in class

$$\langle 0 | \mathbb{T}[\hat{q}(t_1)\hat{q}(t_2)] | 0 \rangle = \langle q(t_1) q(t_2) \rangle = \int_{-\infty}^{\infty} dp_0 \frac{e^{-ip_0(t_1-t_2)}}{p_0^2 - \omega^2 + i\epsilon}$$

For $t_1 > t_2$ complete the contour in the LHP, for $t_1 < t_2$ in the UHP, to get in both cases $\frac{i}{2\omega} e^{-i\omega|t_1-t_2|}$

In particular, $\langle \hat{q}(0)^2 \rangle = \frac{1}{2\omega}$.
The ground state wave-function is $\psi_0 \propto e^{-\omega q^2/2}$, so

$$\langle q^2 \rangle = \frac{\int dq\, q^2 e^{-\omega q^2/2}}{\int dq\, e^{-\omega q^2/2}} = \frac{1}{2\omega}.$$

(b). The energy is $E = \int (\frac{1}{2}\sigma y'^2 + \frac{1}{2}\omega y^2)\, dx$
so $\langle y(x_1) y(x_2) \rangle = \frac{1}{Z}\int [dy]\, y(x_1) y(x_2)\, e^{-\beta E[y]}$

Rescaling $y = \tilde{y}/\sqrt{\sigma\beta}$ we get $\frac{1}{\beta\sigma}\langle \tilde{y}(x_1)\tilde{y}(x_2)\rangle = \frac{1}{\beta\sigma}\int \frac{dp_0\, e^{ip_0(x_1-x_2)}}{p_0^2 + (\frac{\omega}{\sqrt{\sigma}})^2}$

so $\langle y(x_1) y(x_2) \rangle = \frac{1}{2\omega\beta\sqrt{\sigma}} e^{-\frac{\omega}{\sqrt{\sigma}}|x_1-x_2|}$ by contour integration again.

As $\omega \to 0$ this $\to \infty$. However if we calculate

$$\langle (y(x_1) - y(x_2))^2 \rangle = 2\langle y(x)\rangle^2 - 2\langle y(x_1) y(x_2)\rangle$$
$$= \frac{1}{\beta\omega\sqrt{\sigma}}\left(1 - e^{-\frac{\omega}{\sqrt{\sigma}}|x_1-x_2|}\right)$$

this has a finite limit $|x_1 - x_2|/\beta\sigma$.

2. We have in general $\left\langle e^{\int d^d x\, J(x)\phi(x)} \right\rangle = e^{\frac{1}{2\beta J}\int d^d x\, d^d y\, J(x)\Delta(x-y)J(y)}$

where $\Delta(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot(x-y)}}{p^2 + \epsilon^2}$

Taking J as suggested gives us $\langle e^{i\phi(x_1)} e^{-i\phi(x_2)} \rangle = \langle \cos(\phi(x_1) - \phi(x_2)) \rangle$

So $\langle \cos(\phi(x_1)-\phi(x_2))\rangle = e^{\frac{1}{2\beta J}\int d^d x\, d^d y\, (i\delta(x-x_1)-i\delta(x-x_2))\Delta(x-y) \times (i\delta(y-x_1)-i\delta(y-x_2))}$

$= e^{\frac{1}{\beta J}(\Delta(x_1-x_2)-\Delta(0))}$

where $\Delta(x_1-x_2)-\Delta(0) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x_1-x_2)}-1}{p^2+\epsilon^2}$

For $d=1$ we can do this by contour integration $: \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\left(e^{-\epsilon|x_1-x_2|}-1\right)$

so $\langle \cos(\phi(x_1)-\phi(x_2))\rangle = e^{-|x_1-x_2|/2\beta J}$

– paramagnetic behaviour

$d=2$ The integral is convergent at $p=0$ but not as $p\to\infty$.
Impose a cut-off $|p|<\Lambda \sim a^{-1}$ (a is the lattice spacing of the original problem).
The integral is then $F(|x_1-x_2|/a)$.
As $a\to 0$ we have $-\int_{|p|<a^{-1}} \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} \sim -\frac{1}{2\pi}\log a^{-1}$

Hence as $|x_1-x_2|\to\infty$ at fixed a we get $\sim -\frac{1}{2\pi}\log\frac{|x_1-x_2|}{a}$.

So $\langle \cos(\phi(x_1)-\phi(x_2))\rangle \sim e^{-\frac{1}{2\pi\beta J}\log\frac{|x_1-x_2|}{a}} \sim \left(\frac{a}{|x_1-x_2|}\right)^{\frac{1}{2\pi\beta J}}$

– temperature-dependent power law.

$d=3$: we definitely need a cut-off; and $\Delta(x_1-x_2)-\Delta(0) \approx \frac{1}{2\pi^2 a}$

so $\langle \cos(\phi(x_1)-\phi(x_2))\rangle \to$ const. $\Rightarrow |\langle e^{i\phi(x_1)}\rangle|^2$.
The magnet is <u>ordered</u> in this approximation.

3 (a). $1/(k_x^2+k_y^2+k_z^4)$

b). There was some confusion here. We shouldn't take $2\mu^2\phi_1\phi_2$ as part of the interaction (otherwise the question is too easy, apart from anything else). The particle states are eigenstates of the hamiltonian, including the $\phi_1\phi_2$ term

If we write $\mathcal{L}$ as a matrix: $(\phi_1\ \phi_2)\begin{pmatrix} p^2+m_1^2 & \mu^2 \\ \mu^2 & p^2+m_2^2 \end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$

then the propagators are the inverse of this. $\frac{1}{\det}\begin{pmatrix} p^2+m_2^2 & -\mu^2 \\ -\mu^2 & p^2+m_1^2 \end{pmatrix}$

I.e. $\langle\phi_1\phi_1\rangle = \dfrac{p^2+m_2^2}{(p^2+m_1^2)(p^2+m_2^2)-\mu^4}$

$\langle\phi_1\phi_2\rangle = \dfrac{-\mu^2}{(p^2+m_1^2)(p^2+m_2^2)-\mu^4}$ etc.

However, the particle states are the eigenvectors i.e. linear combinations of ϕ_1 & ϕ_2 with propagators

$$\frac{1}{p^2+\dfrac{m_1^2+m_2^2\pm\sqrt{(m_1^2-m_2^2)^2+4\mu^4}}{2}}$$

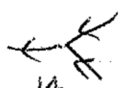
This problem occurs in $K_0, \bar{K}_0$ mesons.

c). If we write $\phi(x,\tau) = \int\frac{d^Dp\,d\omega}{(2\pi)^{D+1}}e^{i\omega\tau}e^{ip\cdot x}\tilde{\phi}(p,\omega)$

$$\mathcal{L} = \int\frac{d^Dp\,d\omega}{(2\pi)^{D+1}}\tilde{\phi}(p,\omega)^*(i\omega+k^2)\tilde{\phi}(p,\omega)$$

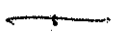
I assume we are in imaginary time (since I used τ not t), so the propagator is the inverse $\dfrac{1}{i\omega+k^2}$.

4). a). Some confusion here: I was assuming that the bare propagator is $\langle\phi\,\phi^*\rangle = \leftarrow$.

In that case the vertices are  and , both -2λ.

b). $-\lambda\cos\phi = -\lambda\left(1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}+\dots\right)$

The factorials get cancelled and we have

 $+\lambda$

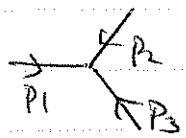
 $-\lambda$

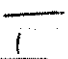
 $+\lambda$, etc.

c). When we write ϕ in terms of its Fourier transform

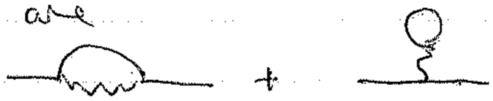
$$\partial_\mu\phi \to ip_\mu\tilde{\phi}(p).$$

so $-\lambda\phi(\partial_\mu\phi)(\partial^\mu\phi) \to -\lambda\tilde{\phi}(p_1)(ip_{2\mu})\tilde{\phi}(p_2)(ip_{3\mu})\tilde{\phi}(p_3)$
$$= \lambda(p_2\cdot p_3)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3).$$

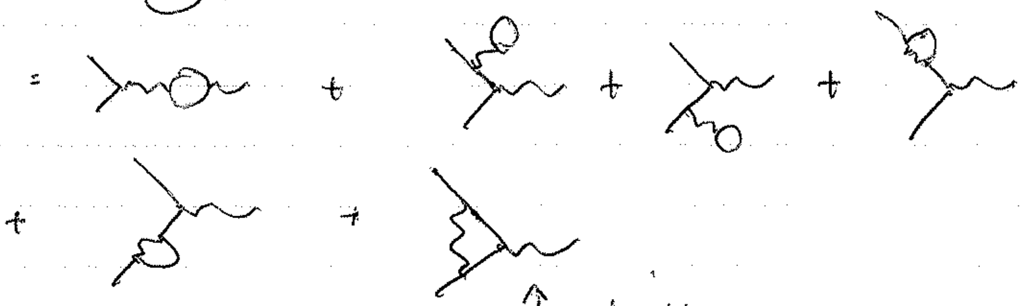
so a vertex like  gives $\lambda[(p_1\cdot p_2)+(p_2\cdot p_3)+(p_3\cdot p_1)]$.
$$= -\frac{\lambda}{2}\left[p_1^2+p_2^2+p_3^2\right] \quad \text{since } \sum p_j = 0.$$

5). Denoting $\langle\phi\,\phi\rangle =$  $= \frac{1}{k^2+m^2}$ $\quad\langle\Phi\,\Phi\rangle =$  $= \frac{1}{k^2+M^2}$ $\quad$  $= -\lambda$

the 1-loop diagrams are

$\langle\phi\,\phi\rangle$: 

$\langle\Phi\,\Phi\rangle$: 

$\langle\phi\,\phi\,\Phi\rangle$ = 

irredecible

The integral for the last diagram is (at $p_1=p_2=p_3=0$)
$$\int\frac{d^dq}{(q^2+M^2)(q^2+m^2)^2} = -\frac{\partial}{\partial m^2}\int\frac{d^dq}{(q^2+M^2)(q^2+m^2)}$$
$$= -\frac{\partial}{\partial m^2}\int_0^1 dx\int\frac{d^dq}{\left(q^2+(1-x)M^2+xm^2\right)^2}$$
$$= -\frac{\partial}{\partial m^2}\left[\frac{\pi^{d/2}}{(2\pi)^d}\Gamma\left(2-\tfrac{d}{2}\right)\int_0^1 dx\left[(1-x)M^2+xm^2\right]^{\frac{d}{2}-2}\right] \quad (\text{as for } $$  $)$.
$$= -\frac{\pi^{d/2}}{(2\pi)^d}\underbrace{\left(\tfrac{d}{2}-2\right)\Gamma\left(2-\tfrac{d}{2}\right)}_{=-\Gamma(3-\frac{d}{2})}\int_0^1 dx\, x\left[(1-x)M^2+xm^2\right]^{\frac{d}{2}-3}$$

The integral over x can actually be done by letting $M^2+x(m^2-M^2)=y$.