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Coherent State Path Integral.

Often $\hat{H}$ does not have the form $\frac{1}{2}\hat{p}^2 + V(\hat{q})$ but depends more generally on annihilation and creation operators $\hat{a}, \hat{a}^\dagger$, with $[\hat{a}, \hat{a}^\dagger] = 1$. For simplicity we consider only one degree of freedom. As usual there is a state $|0\rangle$ such that $\hat{a}|0\rangle = 0$, and the normalised n-particle state is $|n\rangle = ((\hat{a}^\dagger)^n/\sqrt{n!})|0\rangle$.

We want to express the evolution operator $e^{-i\hat{H}(t_f - t_i)/\hbar}$ as a path integral. As before we break up the time interval into subintervals of length Δt and insert complete sets of states at each intermediate time t_j. However this time we insert complete sets of *coherent states* of the form $e^{\phi\hat{a}^\dagger}|0\rangle$, where ϕ is a complex number. The completeness relation for these states is

$$\hat{1} = \int \frac{d^2\phi}{\pi} e^{-\phi^*\phi}\, e\phi\hat{a}^\dagger|0\rangle\langle 0|e^{\phi^*\hat{a}}$$

where $d^2\phi = d\mathrm{Re}\phi\, d\mathrm{Im}\phi$. To check this, expand the rhs:

$$\int \frac{d^2\phi}{\pi} e^{-\phi^*\phi} \sum_{m,n=0}^{\infty} \frac{\phi^m}{m!}\frac{\phi^{*n}}{n!}\, (\hat{a}^\dagger)^m|0\rangle\langle 0|\hat{a}^n$$

Let $\phi = \rho e^{i\theta}$. The angular integration is $\int_0^{2\pi} e^{i(m-n)\theta} d\theta = 2\pi\delta_{mn}$, and the radial integration is then

$$2\int_0^\infty \rho d\rho \rho^{2n} e^{-\rho^2} = n!$$

so we are left with $\sum_{n=0}^\infty |n\rangle\langle n| = 1$.

For the propagation between neighbouring time slices we then have

$$\langle 0|e^{\phi^*(t+\Delta t)\hat{a}}\, e^{-(i\Delta t/\hbar)\hat{H}(\hat{a}^\dagger, \hat{a})}\, e^{\phi(t)\hat{a}^\dagger}|0\rangle \tag{1}$$

Now, in general

$$\begin{aligned} e^{\phi^*\hat{a}}\, F(\hat{a}^\dagger, \hat{a}) &= F(\hat{a}^\dagger + \phi^*, \hat{a})\, e^{\phi^*\hat{a}} \\ F(\hat{a}^\dagger, \hat{a})\, e^{\phi\hat{a}^\dagger} &= e^{\phi\hat{a}^\dagger}\, F(\hat{a}^\dagger, \hat{a} + \phi) \end{aligned}$$

[Can you prove this? Use the fact that $\hat{a}$ acts like $\partial/\partial\hat{a}^\dagger$ and $\hat{a}^\dagger$ like $-\partial/\partial\hat{a}$.]

So, taking the left-hand factor through the expression, (1) becomes

$$\langle 0|e^{-(i\Delta/\hbar)\hat{H}(\hat{a}^\dagger + \phi^*(t+\Delta t), \hat{a})}\, e^{\phi(t)(\hat{a}^\dagger + \phi^*(t+\Delta t))}\, e^{\phi^*(t+\Delta t)\hat{a}}|0\rangle$$

The rightmost exponential gives 1 acting on $|0\rangle$, since $\hat{a}|0\rangle = 0$. Now we take the factor $e^{\phi(t)\hat{a}^\dagger}$ all the way to the left and get

$$\langle 0|e^{\phi(t)\hat{a}^\dagger}\; e^{-(i\Delta/\hbar)\hat{H}(\hat{a}^\dagger+\phi^*(t+\Delta t),\hat{a}+\phi(t))}\; e^{\phi(t)\phi^*(t+\Delta t)}|0\rangle$$

Now the leftmost factor can be set $= 1$. We are left with

$$\langle 0|e^{-(i\Delta/\hbar)\hat{H}(\hat{a}^\dagger+\phi^*(t+\Delta t),\hat{a}+\phi(t))}|0\rangle \quad e^{\phi(t)\phi^*(t+\Delta t)}$$

As long as $\hat{H}$ is normal ordered, that is all factors of $\hat{a}$ are written to the right and all factors of $\hat{a}^\dagger$ to the left, for small Δt we can expand out the first term to get

$$\begin{aligned}
&\langle 0|1 - (i\Delta/\hbar)\hat{H}(\hat{a}^\dagger + \phi^*(t+\Delta t), \hat{a} + \phi(t)) + O((\Delta t)^2)|0\rangle \\
&= 1 - (i\Delta/\hbar)H(\phi^*(t+\Delta t), \phi(t)) + O((\Delta t)^2) \\
&\approx e^{-(i\Delta/\hbar)H(\phi^*(t+\Delta t),\phi(t))}
\end{aligned}$$

The second factor, combined with the measure $e^{-\phi^*(t+\Delta t)\phi(t+\Delta t)}$ gives

$$\prod_j e^{-\phi^*(t_j+\Delta t)\left(\phi^*(t_j+\Delta t)-\phi^*(t_j)\right)} \longrightarrow e^{(i/\hbar)\int dt\phi^*(t)i\hbar\partial_t\phi(t)}$$

Putting together all the pieces, we have

$$\int [d^2\phi] e^{(i/\hbar)\int dt\left[i\hbar\phi^*\partial_t\phi - H(\phi^*,\phi)\right]}$$

Although the path there was arduous, the final result is very simple: just replace the operators $\hat{a}, \hat{a}^\dagger$ in $\hat{H}$ by c-number functions $\phi(t), \phi^*(t)$. You can check that if you write $\hat{p}, \hat{q}$ in terms of $\hat{a}, \hat{a}^\dagger$ then the coherent state path integral reduces to Feynman's.