

The free energy per site $f \equiv -(k_B T/N) \ln Z$ for a given M is therefore, in the mean field approximation,

$$f_{MF}(H) = \frac{1}{2}JM^2 - \beta^{-1} \ln \cosh \beta(JM + H), \quad (1)$$

apart from an irrelevant constant. At this point, the free energy density is not a function of the magnetisation M , rather, it is a function of the external field H . The field H may be thought of as a source for the spin variables $s(r)$ so that $\ln Z(H)$ is the generating functional of connected correlation functions. The magnetisation is given by

$$M(H) = \frac{1}{N} \sum_r \langle s(r) \rangle_H = (\beta N)^{-1} \frac{d}{dH} \ln Z(H) = -\frac{df}{dH}, \quad (2)$$

where the subscript is just a reminder that the expectation value is in the presence of the source. Since $\ln Z(H)$ is a convex function of H , its derivative is monotonic. Therefore, $M(H)$ may be inverted to yield a function $H(M)$ that gives the value of the external field needed to attain a given expectation value M of the spin variables. In our mean field approximation,

$$M = \tanh \beta(JM + H) \implies H = -JM + \beta^{-1} \tanh^{-1} M. \quad (3)$$

The effective potential $\tilde{f}(M)$, also known as the Gibbs free energy, is the minimum energy (per site) within the space of spin configurations that have a given expectation value M . It is obtained via Legendre transform,

$$\tilde{f}(M) = f(H) + H(M)M. \quad (4)$$

Note that $d\tilde{f}/dM = 0$ as desired. The correct variational free energy in the mean field approximation is

$$\tilde{f}_{MF}(M) = -\frac{1}{2}JM^2 - \beta^{-1} \ln \left(\frac{1}{\sqrt{1-M^2}} \right) + \beta^{-1} M \tanh^{-1} M. \quad (5)$$

Notice that the extrema of \tilde{f}_{MF} are given by $M = \tanh(\beta JM)$, which is just eq. (2.8) in the book. To restore a real external magnetic field, simply write H in (3) as $H + H_0$. But for $H_0 = 0$ and small M , \tilde{f}_{MF} has a Taylor expansion

$$\tilde{f}_{MF} = \text{const.} + \frac{1}{2}(k_B T - J)M^2 + \frac{1}{12}\beta^{-1}M^4 + O(M^6). \quad (6)$$

The coefficient of the M^4 term is positive. Depending on the sign of the quadratic term, the shape of the graph is one of those shown in Figure 2.1 in the book.