Notes on large N.

The O(N)-symmetric generalisation of the Ising field theory is based on an N-component field ϕ_j with action

$$S=\int[\tfrac{1}{2}\sum_{j=1}^{N}(\nabla\phi_j)^2+\tfrac{1}{2}m_0^2\sum_{j=1}^{N}\phi_j^2+\lambda_0(\sum_{j=1}^{N}\phi_j^2)^2]d^dr$$

The cases $N=2$ and $N=3$ correspond to the XY and Heisenberg models respectively. The mean field theory for this model yields critical exponents which are independent of N, but the RG below $d=4$ gives N-dependent results, because, as can easily be checked, the OPE coefficients depend on N.

The limit $N\to\infty$ may be rather unphysical but it turns out to be exactly solvable. There are several ways to derive this. The simplest is to observe that, by the central limit theorem, the random variable $\phi^2\equiv\sum_{j=1}^{N}\phi_j^2$ should have a normal distribution in the large N limit. This means in particular that the fourth cumulant $(\phi^2)^2-3\langle\phi^2\rangle\phi^2$ vanishes, so we can replace the last term in the action by $3\lambda_0\langle\phi^2\rangle\sum_{j=1}^{N}\phi_j^2$. This makes the action gaussian, with an effective (mass)^2

$$m_R^2=m_0^2+6\lambda_0\langle\phi^2\rangle$$

This last factor can then be computed self-consistently using the rules of gaussian integration:

$$\langle\phi_j(0)^2\rangle=\int_{|k|<\Lambda}\frac{1}{k^2+m_R^2}\frac{d^dk}{(2\pi)^d}$$

where we have included a UV cut-off. Thus

$$m_R^2=m_0^2+6\lambda_0N\int_{|k|<\Lambda}\frac{1}{k^2+m_R^2}\frac{d^dk}{(2\pi)^d}$$

We see that to get a non-trivial large N limit we actually have to let $\lambda_0\to0$ in such a way that λ_0N is fixed.

The critical point is when $m_R^2=0$, that is

$$m_0^2=m_{0c}^2\equiv-6\lambda_0N\int_{|k|<\Lambda}\frac{1}{k^2}\frac{d^dk}{(2\pi)^d}$$

Note two things about this: (a) the bare (mass)^2 is *negative* at the critical point: this reflects the fact that the real T_c is less than its mean field value, since the fluctuations act to disorder the system; (b) for $d \leq 2$ the integral is IR divergent, so $m_{0c}^2 \to -\infty$ and T_c is pushed all the way to zero. This is an example of the Mermin-Wagner-Coleman theorem that says that a continuous symmetry cannot be spontaneously broken in $d \leq 2$ dimensions.

If we let $t \equiv m_0^2 - m_{0c}^2 \propto T - T_c$ then we have

$$\begin{aligned} m_R^2 &= t + 6\lambda_0 N \int_{|k|<\Lambda} \left(\frac{1}{k^2 + m_R^2} - \frac{1}{k^2} \right) \frac{d^d k}{(2\pi)^d} \\ &= t - 6\lambda_0 N m_R^2 \int_{|k|<\Lambda} \frac{1}{k^2(k^2 + m_R^2)} \frac{d^d k}{(2\pi)^d} \end{aligned}$$

We now see the importance of $d = 4$. For $d > 4$ the last integral is finite as $m_R^2 \to 0$, but it is strongly dependent on Λ, and we find, as $t \to 0$,

$$m_R^2 \sim t - O(m_R^2 \lambda_0 N \Lambda^{d-4})$$

so that $m_R^2 \propto t$. This implies that the correlation length $\xi = m_R^{-1} \propto t^{-1/2}$ so the critical exponent $\nu = \frac{1}{2}$, the mean field value.

For $2 < d < 4$, on the other hand, the integral is UV convergent and we can remove the cut-off. In that case, by dimensional analysis,

$$m_R^2 = t - O(m_R^{d-2} \lambda_0 N)$$

As $m_R \to 0$ the second term on the rhs dominates the lhs so the two terms on the rhs must balance: $m_R \propto t^{1/(d-2)}$, so

$$\nu = 1/(d-2)$$

There are interesting logarithmic corrections for $d = 4$.

It is possible to use the large N limit as the basis for a systematic $1/N$ expansion of the critical behaviour, which complements the ϵ-expansion.

Large N limits are also very important in systems with other symmetries, for example SU(N), but they tend to be much more difficult to analyse.