11
Conformal symmetry

We saw in Chapter 3 that the hamiltonian for a system at a critical point flows under the renormalization group into a critical fixed point. Under a renormalization group transformation, the microscopic length scale is rescaled by a constant factor b, and so the coordinates of a given point, as measured in units of this length scale, transform according to $\mathbf{r} \to b^{-1}\mathbf{r}$. This is called a *scale transformation.* Once the flows reach such a fixed point, the parameters of the hamiltonian no longer change, and it is said to be *scale invariant.* As well as being scale invariant, the fixed point hamiltonian usually possesses other spatial symmetries. For example, if the underlying model is defined on a lattice, so that its hamiltonian is invariant under lattice translations, the corresponding critical fixed point hamiltonian is generally invariant under arbitrary uniform translations. This is because terms which might be added to the hamiltonian which break the symmetry under continuous translations down to its subgroup of lattice translations are irrelevant at such a fixed point. Similarly, if the lattice model is invariant under a sufficiently large subgroup of the rotation group (for example, if the interactions in the x and y directions on a square lattice are equal), then the fixed point hamiltonian enjoys full rotational invariance. As discussed on p.**??**, even if the interactions are anisotropic, rotational invariance may often be recovered by a suitable finite relative rescaling of the coordinates. For systems with intrinsic anisotropy, this is not the case, and we shall not discuss such cases further in this chapter. These operations of translation, rotation and scaling (dilatation) form a group. Under a general element of this group an arbitrary correlation function of scaling operators transforms in a simple way at the fixed point:

$$\langle \phi_1(r_1)\phi_2(r_2)\ldots\rangle = \prod_j b^{-x_j}\langle \phi_1(r_1')\phi_2(r_2')\ldots\rangle, \qquad (11.1)$$

where x_j is the scaling dimension of ϕ_j. In writing this, we have assumed that all the operators are scalars under rotation. Otherwise the appropriate rotation matrices need to appear on the right hand side.

It turns out, however, that as long as the fixed point hamiltonian contains only short range interactions, it is invariant under the larger symmetry of *conformal transformations*. For our purposes, a conformal transformation $r \to r'$ is one which *locally* corresponds to a combination of a translation, rotation and dilatation.† This is simpler to illustrate for the case of an infinitesimal transformation

$$r^\mu \to r'^\mu + \alpha^\mu(r), \tag{11.2}$$

where $\alpha^\mu(r) \ll 1$. If α^μ is a constant, this is of course simply a translation. When $\alpha^\mu(r)$ is slowly varying, the matrix of derivatives $\alpha^\mu{}_\nu = \partial\alpha^\mu/\partial r^\nu$ may be written as a sum of three pieces:‡

- an antisymmetric part $\alpha^{\mu,\nu} - \alpha^{\nu,\mu}$, which corresponds locally to a rotation;
- a diagonal part $\alpha^\lambda{}_{,\lambda} g^{\mu\nu}$, corresponding to a dilatation; and
- a traceless symmetric part $\alpha^{\mu,\nu} + \alpha^{\nu,\mu} - (2/d)\alpha^\lambda{}_{,\lambda} g^{\mu\nu}$, which may be thought of as the components of the local *shear*.

Conformal transformations are those for which this last piece vanishes. Since they have no shear component, they possess the property of preserving the angles between the tangents to curves meeting at a given point. An example of such a transformation in two dimensions is shown in Figure 12.1. The heuristic argument that invariance of the fixed point hamiltonian under translations, rotations and dilatations should imply its invariance under this larger set of symmetry transformations is deceptively simple. Imagine performing an inhomogeneous renormalization group transformation from the original regular lattice to one which is distorted by such a conformal mapping. In terms of a block spin transformation, this would mean replacing all the original degrees of freedom

† More correctly, conformal transformations should be viewed as acting on the *metric*, in such a way that $g_{\mu\nu} \to \Omega(r) g_{\mu\nu}$. This allows for conformal mappings between flat and curved spaces, for example. However, we shall restrict ourselves to flat spaces as these are more appropriate for statistical mechanics.

‡ Here, and throughout this chapter, we use the summation convention.

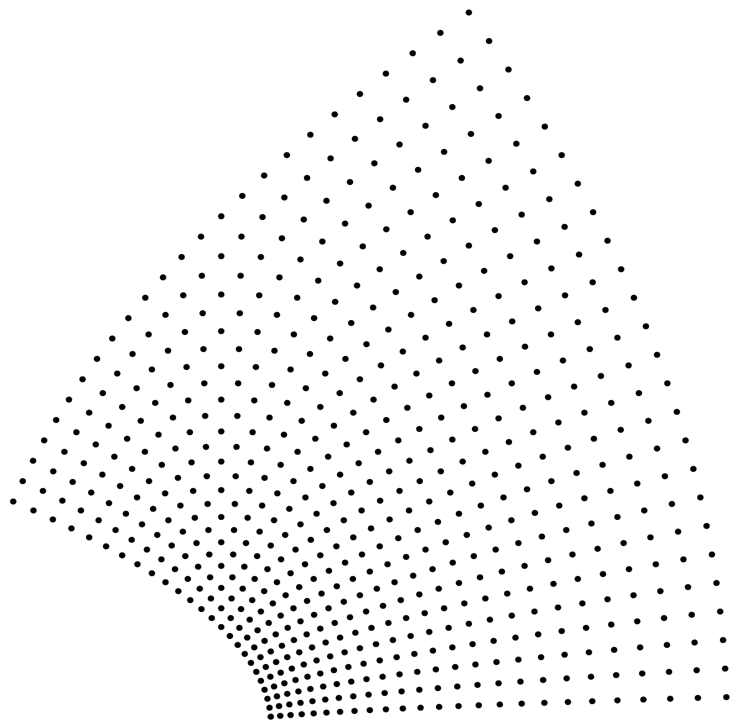

Figure 11.1. Example of a conformal transformation of part of a square lattice.

inside one cell of Figure 12.1 by a single block spin. In the vicinity of a given cell, the lattice spacing is rescaled by a factor $b(r)$, where $b(r)^{-d}$ is the Jacobian of the transformation $r \to r'$. If $b(r)$ is sufficiently slowly varying, then the way in which the local parameters of the hamiltonian transform in the neighbourhood of this cell will be just as if we were performing a uniform renormalization group transformation with rescaling factor $b(r)$ everywhere. Since the fixed point hamiltonian is invariant under such transformations, it is also invariant in the more general case when b varies with r. This argument can apply, of course, only if the fixed point hamiltonian is of sufficiently short range.

A simple generalisation of the reasoning in Section **??** which led to (12.1) now yields the transformation law for correlation functions

$$\langle\phi_1(r_1)\phi_2(r_2)\ldots\rangle = \prod_j b(r_j)^{-x_j}\langle\phi_1(r_1')\phi_2(r_2')\ldots\rangle, \qquad (11.3)$$

once again for scalar operators. Clearly the above argument is heuristic at best, and, in Section 12.3, we shall put it on a more systematic footing. In fact, it will be seen that (12.3) cannot be true for all scalar scaling operators. Indeed, it is a simple exercise to show that if it is true for the correlation functions of ϕ, it cannot be valid for its derivatives, for example $\nabla^2\phi$, even though these behave correctly under scale transformations. Instead, as we shall show, the transformation law (12.3) holds for a restricted class of scaling operators called *primary*. However, fortunately, the operators corresponding to the most relevant scaling variables are usually of this type.

11.1 Conformal transformations

The condition that an infinitesimal transformation be conformal is

$$\alpha^{\mu,\nu} + \alpha^{\nu,\mu} - (2/d)\alpha^{\lambda}{}_{,\lambda}g^{\mu\nu} = 0, \qquad (11.4)$$

These equations are very restrictive when $d > 2$. In fact, the only solutions in that case, apart from infinitesimal translations, rotations and dilatations, are the so-called *special conformal transformations*

$$\alpha^{\mu}(r) = b^{\mu}r^2 - 2(b^{\lambda}r_{\lambda})r^{\mu}. \qquad (11.5)$$

These may be thought of as made up of a finite conformal transformation, the *inversion* mapping $r^{\mu} \to r'^{\mu} = r^{\mu}/r^2$, followed by an infinitesimal translation by the vector b^{μ}, then a further inversion. Thus the conformal transformations for $d > 2$ may be generated by adding the discrete operation of inversion to the other three.

In two dimensions, however, there is far greater freedom. This may be seen most simply if we write (12.4) using *complex coordinates*, defined by $z = r^1 + ir^2$, $\bar{z} = r^1 - ir^2$. For most purposes, we may disregard the fact that $\bar{z}$ is the complex conjugate of z, and treat them as if they were independent complex variables. The line element in this coordinate system is

$$ds^2 = (dr^1)^2 + (dr^2)^2 = dzd\bar{z}, \qquad (11.6)$$

so that the metric is no longer diagonal. In fact

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{11.7}$$

For this reason, we must distinguish upper and lower indices. For example, for a vector $b_z = \frac{1}{2}b^{\bar{z}}$ and $b_{\bar{z}} = \frac{1}{2}b^z$. In these coordinates, the $(z\bar{z})$ and $(\bar{z}z)$ components of (12.4) are trivially satisfied for $d = 2$, and the others reduce to

$$\alpha^z{}_{,\bar{z}} = \alpha^{\bar{z}}{}_{,z} = 0. \tag{11.8}$$

Thus α^z depends only on z, rather than $\bar{z}$, which means that it is an *analytic* function of z. Similarly, $\alpha^{\bar{z}}$ is an analytic function of $\bar{z}$. This is the well known result that analytic functions correspond to conformal transformations in $d = 2$, and is the reason such mappings are so useful for solving Laplace's equation.

The notion of complex coordinates also makes it rather simple to discuss non-scalar operators in two dimensions. In general, we may classify such operators according to their *spin*.† Under a rotation $z \to ze^{i\theta}$, an operator of spin s transforms by a factor $e^{is\theta}$. What this means is that its two-point correlation function, for example, behaves like

$$\langle\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = |z_{12}|^{-2x}(\bar{z}_{12}/z_{12})^s, \tag{11.9}$$

where $z_{12} = z_1 - z_2$ and x is the usual scaling dimension of ϕ. Note that, for this two-point function to be single-valued, $2s$ should be an integer. (12.9) suggests that we define the so-called *complex scaling dimensions* $(h,\bar{h})$ by $x = h + \bar{h}$ and $s = h - \bar{h}$, so that the two-point function may be written $z_{12}^{-2h}\bar{z}_{12}^{-2\bar{h}}$. Note that $\bar{h}$ is *not* the complex conjugate of h. In fact, they are both real numbers. A simple consequence of this classification is that the operator product expansion of Section **??** has a simple form in $d = 2$, even for non-scalar operators:

$$\phi_i(z_1,\bar{z}_1)\cdot\phi_j(z_2,\bar{z}_2) = \sum_k c_{ijk}\, z_{12}^{-h_i-h_j+h_k}\; \bar{z}_{12}^{-\bar{h}_i-\bar{h}_j+\bar{h}_k}\; \phi_k(z_2,\bar{z}_2). \tag{11.10}$$

Similarly, the transformation law (12.3) for correlation functions under a conformal transformation corresponding to the analytic

† This has no physical connection with the quantum mechanical idea of spin.

mapping $z \to z' = w(z)$ may be written

$$\langle \phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle = \prod_i w'(z_i)^{h_i}\overline{w'(z_i)}^{\bar h_i}\,\langle \phi_1(z_1',\bar z_1')\phi_2(z_2',\bar z_2')\ldots\rangle, \tag{11.11}$$

since the local dilatation factor is $|w'(z)|^{-1}$ and the local rotation is $\arg w'(z)$.

11.2 Simple consequences of conformal symmetry

In this section we shall assume the correctness of the transformation law (12.3) and deduce some simple consequences. The first set of results is valid for arbitrary dimension, since it exploits the symmetry under only special conformal transformations. For simplicity we then restrict the considerations to scalar operators.

Consider first the two-point correlation function of two different operators $\langle\phi_1(r_1)\phi_2(r_2)\rangle$. Conformal symmetry implies that this vanishes unless the scaling dimensions x_1 and x_2 are equal. The essence of the argument is simple. We can always choose a conformal transformation which maps the points r_1 and r_2 into, say, r_1' and r_2' respectively, under which the two-point function will be multiplied by a factor $b(r_1)^{-x_1}b(r_2)^{-x_2}$. Now imagine making the same transformation on $\langle\phi_2(r_1)\phi_1(r_2)\rangle$, where the two operators have been exchanged. This cannot affect the value of the correlation function, since they are related by a rotation through 180°. But now the rescaling factor will be $b(r_1)^{-x_2}b(r_2)^{-x_1}$. Since $b(r_1)\neq b(r_2)$ for a conformal mapping, the only way for these two results to agree when $x_1\neq x_2$ is for the two-point function itself to vanish.

The above argument does not work when $x_1 = x_2$, so that if we consider the set of all operators ϕ_i with the same scaling dimension x, their two-point functions have the general form $\langle\phi_i(r_1)\phi_j(r_2)\rangle = d_{ij}r_{12}^{-2x}$. However, since d_{ij} must be real and symmetric, we may choose suitable linear combinations of the ϕ_i so that it is diagonal. In models satisfying *reflection positivity*,† (for

† This is true of most fixed points describing models with positive Boltzmann weights. Even microscopic models whose transfer matrix is not symmetric may correspond, at the fixed point, to a reflection positive theory. The main exceptions are cases like the $O(n)$ and Q-state Potts models for non-positive

example, when their transfer matrix may be brought into a symmetric form), these diagonal elements are all positive, and so, by normalising the operators appropriately, d_{ij} is simply δ_{ij}. Thus one of the simple consequences of conformal invariance is the *orthogonality* of scaling operators, in the sense that their two-point functions may be taken to have the form

$$\langle \phi_i(r_1)\phi_j(r_2)\rangle = \frac{\delta_{ij}}{r_{12}^{2x_i}}. \tag{11.12}$$

While such results sometimes also follow from the internal symmetries of the model (for example, the energy-magnetisation two-point function in the Ising model vanishes anyway on the grounds of the symmetry of the fixed point hamiltonian under reversing all the spins), we see that their provenance is more general.

For the three-point functions, conformal invariance completely fixes their functional dependence. To see this, note that, by translations, rotations and dilatations alone, two arbitrary points r_1 and r_2 may be mapped to two pre-assigned points. This is the reason why these symmetries are sufficient to fix the functional form of the two-point functions. The special conformal transformations then give one additional relation whereby *three* arbitrary points r_1, r_2 and r_3 may be mapped to three preassigned points r_1', r_2' and r_3'. Thus the three-point function $\langle\phi_i(r_1)\phi_j(r_2)\phi_k(r_3)\rangle$ may be related to the same correlation function with $r_i \to r_i'$, with the dependence on the r_i entering solely through the scaling factors $\prod_i b(r_i)^{x_i}$. The algebraic details of this calculation are not particularly illuminating, and it is simpler to verify the result, which has the remarkable elegant form

$$\frac{C_{ijk}}{|r_1-r_2|^{x_i+x_j-x_k}|r_2-r_3|^{x_j+x_k-x_i}|r_3-r_1|^{x_k+x_i-x_j}}, \tag{11.13}$$

where C_{ijk} is a constant. In fact, this is equal to the operator product expansion coefficient c_{ijk} defined in Section **??**, as long as the operators are correctly normalised as in (12.12) above. This follows immediately if we take the correlation function of both sides of the operator product expansion in (12.10) with the operator ϕ_k. Since we are free to permute the points in (12.13) without changing the value of the three-point function, it follows that the

integer n or Q, described in Sections **??** and **??**.

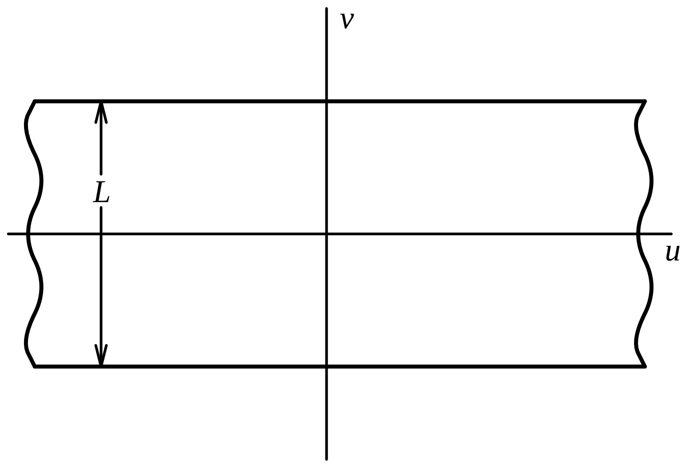

Figure 11.2. Finite-width strip geometry.

C_{ijk}, and therefore the operator product expansion coefficients (in the orthonormal operator basis) are totally symmetric functions of the indices (ijk). This is another very powerful result of conformal symmetry.

In two dimensions, conformal symmetry is much more powerful, because any analytic function $w(z)$ gives a conformal mapping $z \to z' = w(z)$. However, such a transformation will not, in general, map the plane onto itself, and so it is important to realise, in writing the transformation law (12.1), the correlation functions on either side may be evaluated in different geometries.† Sometimes this fact may be exploited, as when conformal mappings are used to transform a solution of Laplace's equation in one geometry to that in a simpler geometry.

Consider, for example, the mapping given by $w = (L/2\pi)\ln z$. This is analytic everywhere except at the origin, and maps the whole complex z-plane (minus this point) into the strip $|\mathrm{Im} w| \leq L/2$. A function which is single-valued in the plane will satisfy periodic boundary conditions between opposite edges of the strip (see Figure 12.2). This is an example of a quasi-one-dimensional finite size geometry discussed in Section **??**. The form of the two-point

† The special conformal transformations do preserve the plane with the point at infinity added (the Riemann sphere).

function in the strip with periodic boundary conditions then follows from the transformation law (11.11) and its form in the plane, which, restricting to the scalar case for simplicity, is $|z_{12}|^{-2x}$. The result is

$$\frac{(2\pi/L)^{2x}}{\left[2\cosh\left(2\pi(u_1-u_2)/L\right)-2\cos\left(2\pi(v_1-v_2)/L\right)\right]^x}, \tag{11.14}$$

where $w = u + iv$, so that u and v are Cartesian coordinates running along and across the strip respectively, as shown in Figure 12.2. When $|w_{12}| \ll L$, this behaves as $|w_{12}|^{-2x}$, independent of the finite width L, but, for $|u_1 - u_2| \gg L$, the correlation function decays exponentially

$$\langle\phi(u_1,v_1)\phi(u_2,v_2)\rangle \sim \left(\frac{2\pi}{L}\right)^{2x} e^{-(2\pi x/L)|u_1-u_2|}. \tag{11.15}$$

Such an exponential decay is to be expected in a quasi-one-dimensional geometry. From (12.15) may be inferred the correlation length along the strip

$$\xi = L/(2\pi x). \tag{11.16}$$

The fact that the result is proportional to L is a consequence of finite-size scaling (see Section **??**.) However, what is remarkable about (12.16) is that the amplitude ξ/L is simply related to the scaling dimension x. This prediction of conformal invariance in two dimensions has been amply verified by numerical and exact studies, and is now an important tool for extracting the scaling dimensions of otherwise unsolvable models. This is because the correlation length ξ is given in terms of the eigenvalue λ_j of the transfer matrix acting along the strip by the formula $\xi^{-1} = -\ln(\lambda_i/\lambda_0)$, where λ_0 is the largest, and λ_i is the dominant subleading eigenvalue which couples to the operator in question. For finite L, the transfer matrix is usually finite-dimensional, and is amenable to exact diagonalisation. In fact, since (12.15) is true for every (primary) operator, it follows that each scaling dimension x of the fixed point theory corresponds to an eigenstate of the transfer matrix. Actually, a stronger statement is true: there is a one-to-one correspondence between the full set of scaling operators at the fixed point, and the eigenvectors of the transfer matrix on the strip (at least those whose eigenvalues scale like L^{-1} as $L \to \infty$).

11.3 The stress tensor

A number of similar results follow from the transformation law (11.11), but, in order to understand better its theoretical underpinnings, it is necessary to discuss the crucial role played by the *stress tensor.* This is a special scaling operator, which may be introduced as follows.† Suppose, instead of making a rescaling of the lattice which corresponds to a conformal transformation, that is, corresponds locally to a rotation and dilatation, we allow in addition the possibility of a shear component. If we imagine constructing a renormalization group transformation to this distorted lattice, there is no longer any reason to suppose that the fixed point hamiltonian will remain invariant. Instead, it will acquire an additional piece $\delta\mathcal{H}$, which, at least for an infinitesimal transformation, should be expressible as a linear combination of the complete set of scaling operators at the fixed point. If we consider the distortion of the lattice as corresponding to a general infinitesimal coordinate transformation $r^\mu \to r'^\mu + \alpha^\mu(r)$, the change in the hamiltonian may then be written

$$\delta\mathcal{H} = -S_d^{-1}\int T_{\mu\nu}(r)\partial^\mu\alpha^\nu(r)d^dr. \qquad (11.17)$$

This equation, for arbitrary $\alpha^\nu(r)$, then *defines* the stress tensor $T_{\mu\nu}$. The factor of S_d is the area of a unit hypersphere, and is conventionally introduced since it leads to greater convenience in later formulas. Once again, the crucial assumption of short range interactions has been invoked in writing (12.17), as it is implicitly assumed that $\delta\mathcal{H}$ may be written as a sum (integral) over local contributions, each of which depends only on the local distortion. Since, however, $\delta\mathcal{H}$ vanishes whenever this has components only of rotation and dilation, that is when the traceless symmetric part of $\partial^\mu\alpha^\nu$ vanishes, we see that $T_{\mu\nu}$ must be both symmetric and traceless itself.

It is important to realise that the stress tensor is a scaling operator just like the local energy density or magnetisation, although, unlike these, it is not a rotational scalar. As with these operators, it may be expressed as a linear combination of lattice operators expressed in terms of the fundamental degrees of freedom in the

† In quantum field theory, it is often known as the 'improved' energy-momentum tensor.

hamiltonian. For example, consider the critical Ising model on a square lattice, with equal nearest neighbour interactions K in the x and y directions. Suppose we make an infinitesimal shear transformation $x \to x' = (1+\epsilon)x$, $y \to y' = (1-\epsilon)y$, thus distorting the lattice. In terms of the new coordinates, the correlations will now be anisotropic. But we know, following the discussion on p.**??**, that this anisotropy may be accounted for by introducing anisotropic interactions $K_x \neq K_y$. In this case, we should take $K_x = K(1-\lambda\epsilon)$ and $K_y = K(1+\lambda\epsilon)$, where λ is some (non-universal) constant. This generates a new term proportional to a sum over $s(x,y)s(x+a,y) - s(x,y)s(x,y+a)$ in the hamiltonian. On the other hand, for this particular transformation, we see from (12.17) that $\delta\mathcal{H}$ is given by an integral over $T_{xx} - T_{yy}$. Hence, for this model,

$$\begin{aligned} T_{xx} - T_{yy} \propto s(x,y)s(x+a,y) - s(x,y)s(x,y+a) \\ + s(x,y)s(x-a,y) - s(x,y)s(x,y-a), \quad (11.18) \end{aligned}$$

where we have antisymmetrised so that both sides have the same behaviour under reflections. It should be stressed that, in writing (12.18), as with all lattice identifications of scaling operators, it is valid only in the sense that correlation functions of either side are asymptotically the same when the points are far apart. At smaller separations, there are additional, less relevant, operators on the left hand side which will give rise to corrections.

From now on in this section, we shall consider only the case $d = 2$, where the use of complex components considerably simplifies the analysis. Why should one introduce a more general transformation if the aim is to analyse those which are conformal? The reason is that an analytic function $\alpha(z)$ cannot, in general, be small everywhere in the plane, but only in some finite region. Transformations which are infinitesimal everywhere must therefore be non-conformal at some points. Suppose we are interested in the effect of conformal transformations in the vicinity of the origin. Surround this by two regions $|z| < R_1$ and $R_1 < |z| < R_2$ (see Figure 12.3). Now make an infinitesimal transformation $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(r)$ which is everywhere differentiable, and corresponds to the conformal transformation $z \to z' = z + \alpha(z)$ in the first region, while it reduces to the identity $r' = r$ for $|z| > R_2$. Otherwise, it is arbitrary in the annulus in between. The change

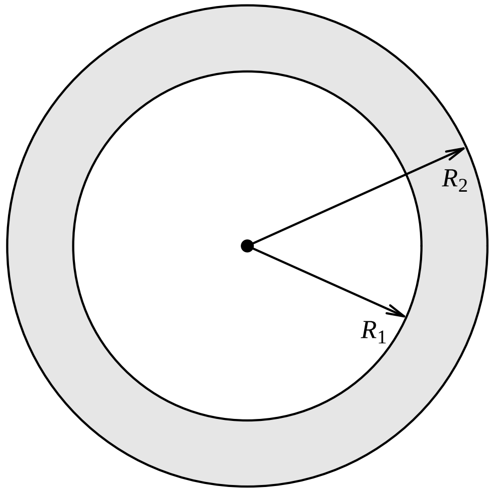

Figure 11.3. Geometry used to study the effect of a localised conformal transformation.

in the hamiltonian is, by (12.17), $\delta\mathcal{H} = -(1/2\pi)\int T_{\mu\nu}\partial^\mu\alpha^\nu d^2r$, where the integrand is non-vanishing only within the annulus. Integrating by parts then gives a term proportional to $\int \alpha^\nu\partial^\mu T_{\mu\nu}d^2r$, together with a surface term $\int \alpha^\nu T_{\mu\nu}dS^\mu$ on each circle. Since α^ν is quite arbitrary within the annulus, and since the result we are going to obtain from this calculation cannot depend on its precise form, then $\partial^\mu T_{\mu\nu}$ must vanish identically. Thus the stress tensor is also *conserved*. The surface term from $|z| = R_2$ vanishes, since $\alpha^\nu = 0$ there.

The stress tensor, being symmetric, has complex components $T_{zz} \equiv T$, $T_{\bar z\bar z} \equiv \overline{T}$ and $T_{z\bar z} = T_{\bar z z}$. In fact, the latter components vanish, since the trace is $T_z^z + T_{\bar z}^{\bar z} = 2T_{\bar z z} + 2T_{z\bar z}$. The conservation conditions then imply that

$$\partial^\mu T_{\mu z} = \partial^z T_{zz} = 2\partial_{\bar z}T = 0 \tag{11.19}$$

$$\partial^\mu T_{\mu\bar z} = \partial^{\bar z} T_{\bar z\bar z} = 2\partial_z\overline{T} = 0. \tag{11.20}$$

This means that correlation functions of $T(z,\bar z)$ in fact depend only on z, so are therefore holomorphic functions. Similarly correlation functions of $\overline{T}$ are antiholomorphic.

In this notation, the boundary term in $\delta\mathcal{H}$ may be written as a contour integral around the circle $C : |z| = R_1$. After a little algebra, this may be written

$$\delta\mathcal{H} = \frac{1}{2\pi i}\int_C \alpha(z)T(z)dz - \frac{1}{2\pi i}\int_C \overline{\alpha(z)}\,\overline{T}(\bar z)d\bar z. \tag{11.21}$$

Now suppose that $\alpha(z) = \epsilon z$, corresponding to dilatation (and a rotation if ϵ is complex). Consider the transformation properties of the correlation function $\langle\phi(0)\ldots\rangle$ of a scaling operator ϕ at the origin, with other operators, represented by the dots, all of whose arguments lie outside the larger circle $|z| = R_2$. Since the transformation near the origin is a pure rotation and dilatation, we know how $\phi(0)$ will transform. On the other hand, the fixed point hamiltonian $\mathcal{H}^*$ changes by $\delta\mathcal{H}$. We may therefore write

$$\langle\phi(0)\ldots\rangle_{\mathcal{H}^*} = (1+\epsilon)^h(1+\bar\epsilon)^{\bar h}\langle\phi(0)\ldots\rangle_{\mathcal{H}^*+\delta\mathcal{H}}, \tag{11.22}$$

where $(h,\bar h)$ are the complex scaling dimensions of ϕ. The correlation function on the right hand side is to be evaluated with respect to the perturbed hamiltonian $\mathcal{H}^* + \delta\mathcal{H}$. This may also be written $\langle\phi(0)\ldots e^{-\delta\mathcal{H}}\rangle_{\mathcal{H}^*}$. Expanding this exponential and equating coefficients of ϵ and $\bar\epsilon$ in (12.22), we then find

$$\frac{1}{2\pi i}\int_C zT(z)\phi(0)dz = h\phi(0), \tag{11.23}$$

with a similar equation involving $\overline{T}$ and $\bar h$.

This has immediate consequences for the operator product expansion of $T(z)$ and $\phi(0)$. T itself has scaling dimensions $(2,0)$ since, from its definition (12.17), it has overall scaling dimension $x = 2$, and does not depend on $\bar z$, so $\bar h = 0$. The operator product expansion must be an analytic function of z, except possibly at $z = 0$, so may involve only integer powers of z:

$$T(z)\phi(0) = \sum_n z^{-2+n}\phi^{(n)}(0), \tag{11.24}$$

thus defining the scaling operators $\phi^{(n)}$. Substituting this into the contour integral in (12.23) implies, by the residue theorem, that $\phi^{(0)} = h\phi$. An analogous argument, taking this time $\alpha = \text{const.}$, corresponding to a uniform translation, similarly gives $\phi^{(1)} = \partial_z\phi$.

Now note that the scaling dimension of $\phi^{(n)}$ is $h+n$. Since we do not expect to find scaling operators with arbitrarily negative dimensions, these $\phi^{(n)}$ must vanish for sufficiently large negative n. The special class of operators for which $\phi^{(n)} = 0$ for *all* $n < 0$

are called *primary*. For these, the leading terms in the operator product expansion with T are therefore already determined by their behaviour under translations, rotations and dilatations to be

$$T(z)\phi(0) = \frac{h}{z^2}\phi(0) + \frac{1}{z}\partial_z\phi(0) + \cdots. \tag{11.25}$$

For such operators we now have a much stronger result. Consider an arbitrary infinitesimal conformal transformation, parametrised by $\alpha(z)$. By analogy with the above argument, the change in $\phi(0)$ is then given by the contour integral

$$\delta\phi(0) = \frac{1}{2\pi i}\int_C \alpha(z)T(z)\phi(0)dz - \frac{1}{2\pi i}\int_C \overline{\alpha(z)}\overline{T}(\bar{z})\phi(0)d\bar{z}. \tag{11.26}$$

Inserting the operator product expansion (12.25), the right hand side may be evaluated by Cauchy's theorem to give

$$\phi(0) \to \phi\left(\alpha(0), \bar{\alpha}(0)\right) + \left(h\alpha'(0) + \bar{h}\bar{\alpha}'(0)\right)\phi(0), \tag{11.27}$$

that is, the transformation properties of a primary operator under conformal transformations are already determined by its behaviour under rotations and dilatations. This result is simple to generalise to the case of a finite conformal transformation $z \to z' = w(z)$:

$$\phi(z,\bar{z}) \to w'(z)^h \overline{w'(z)}^{\bar{h}} \phi(z', \bar{z}'), \tag{11.28}$$

from which follows the transformation law for correlation functions (11.11) already exploited in the previous section.

Special conformal transformations, on the other hand, correspond to taking $\alpha(z) \propto z^2$, and the condition that an operator behave appropriately under this restricted class is that $\phi^{(-1)} = 0$. Such operators are called *quasi-primary.* Roughly speaking, such operators cannot be written as the derivatives of other operators with respect to the coordinates. These considerations extend, in fact, to dimensions $d > 2$. The results of Section 12.2 for the two- and three-point functions then rely only on the assumption that the operators involved are quasi-primary. Any scaling operator is either itself quasi-primary or may be written in terms of derivatives thereof.

It is important to realise that the stress tensor itself is not primary. In fact, its operator product expansion with itself must have the form

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial_z T + \cdots. \quad (11.29)$$

The coefficient of z^{-2} term reflects the fact that T has $h = 2$. There is no z^{-3} term by symmetry under $z \to -z$. The z^{-4} term must be present so that, on taking the expectation value of both sides of the equation, the two-point function $\langle T(z)T(0)\rangle \propto 1/z^4$ is non-zero. Notice that, unlike other scaling operators, we are not free to adjust the normalisation of T so that its two-point function has coefficient unity. This is because this is already fixed by the definition of T through (12.17).

Equation (12.29) thus introduces what turns out to be a ubiquitous property of a fixed point theory in two dimensions, the so-called *conformal anomaly number*, or equivalently the *central charge*, c.† The additional term in (12.29) means that T does not transform in such a simple way as (12.28) under a finite conformal transformation. Instead, there is an additional term

$$T(z) \to w'(z)^2 T(z') - \tfrac{c}{12}\{z', z\}, \quad (11.30)$$

where $\{z', z\} \equiv (w'''w' - \frac{3}{2}w''^2)/w'^2$ is called the Schwartzian derivative.

Consider, for example, the mapping $w(z) = (L/2\pi)\ln z$ from the plane to the strip with periodic boundary conditions. In the plane, $\langle T\rangle = \langle\overline{T}\rangle = 0$ by rotational invariance. Equation (12.30) then yields the corresponding quantity in the strip

$$\langle T\rangle_{\mathrm{strip}} = \langle\overline{T}\rangle_{\mathrm{strip}} = \frac{c}{24}\left(\frac{2\pi}{L}\right)^2. \quad (11.31)$$

This gives the reduced free energy per unit length of the strip

$$E_0(L) = -\frac{1}{2\pi}\int_0^L \langle T_{uu}\rangle dv = -\frac{L}{2\pi}\left(\langle T\rangle_{\mathrm{strip}} + \langle\overline{T}\rangle_{\mathrm{strip}}\right). \quad (11.32)$$

To see this, recall the definition (12.17) of $T_{\mu\nu}$, and consider the transformation $(u, v) \to ((1+\epsilon)u, v)$. The hamiltonian changes by an amount $\delta\mathcal{H} = -\frac{\epsilon}{2\pi}\int T_{uu}\,du\,dv$. The expectation value of this must be balanced by an explicit change in the free energy

† The first name originates from its role when the theory is defined on a curved background, when the trace T^μ_μ is non-zero and equal to $-cR/12$, where R is the scalar curvature. The second comes from the operator formulation of conformal symmetry, where it appears as the coefficient of the central term in the Virasoro algebra.

$-\epsilon E_0(L)$ per unit length. From (12.31), the free energy per unit length of the system in the strip geometry is therefore

$$E_0(L) = -\frac{\pi c}{6L}. \qquad (11.33)$$

Note that in writing this, we have already subtracted off an extensive $O(L)$ term proportional to the bulk free energy. Once again, the L-dependence of (12.33) is as expected from finite-size scaling, but the remarkable result is that its amplitude is related to the central charge of the fixed point theory. Since E_0 is simply given by the logarithm of the largest eigenvalue λ_0 of the transfer matrix, it may be extracted simply from numerical studies. This gives a direct way of measuring c.

A more physical interpretation of the central charge may be made if we consider the coordinate v across the strip as being imaginary time, and u as representing one-dimensional space. As discussed in Section **??**, the partition function in this geometry may be viewed as that for a one-dimensional quantum system, at finite temperature $T = (k_B L)^{-1}$. Equation (12.33) then gives the reduced free energy per unit length $F/(k_B T)$ of this quantum system. From this we may read off the specific heat

$$C \sim \frac{\pi c k_B^2}{3} T. \qquad (11.34)$$

Since this result assumed rotational invariance of the equivalent two-dimensional model, it is valid only for systems with a dynamic exponent $z = 1$, that is, a linear dispersion law $\omega \sim v|k|$ for the elementary excitations. (12.34) is then written in units where $v = 1$. In addition, it is valid only when the width of the strip is much larger than the lattice spacing, which translates into low temperatures. With these provisos, we see that the low temperature behaviour of the specific heat of such a system is linear in T, and its slope is directly related to the value of c. It is instructive to compute the specific heat for free relativistic bosons in one dimension, using standard methods of statistical mechanics. This yields the form (12.34), with $c = 1$. In general, n types of noninteracting bosons would have $c = n$. The central charge may therefore be thought of as counting the number of gapless degrees of freedom of the theory. This interpretation must not be taken too literally, however, since interacting critical theories typically have non-integral values of c!

11.4 Further developments

Beyond this point, the study of conformal symmetry in two dimensions becomes increasingly mathematical and goes beyond the modest scope of this book. It is worth, however, recording some of the major results of the analysis.

As with any continuous symmetry in physics, the generators of conformal transformations form an algebra, called in this instance the Virasoro algebra. There are an infinite number of generators L_n, one for each term in the Laurent expansion of $\alpha(z) = \sum_n a_n z^{-n+1}$, and their commutator algebra has the form

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{c}{12}n(n^2-1)\delta_{n,-m}. \qquad (11.35)$$

In any particular fixed point theory, the L_n are represented by operators acting on the physical space of states. This is constructed by a technique known as radial quantisation. With this method, there is then a one-to-one correspondence between the scaling operators at the fixed point, and those states of the Hilbert space which are eigenstates of the operator L_0. This operator generates dilatations $\alpha(z) \propto z$, and its eigenvalues are just the scaling dimensions. The algebra allows one to organise the eigenstates of L_0, and hence the scaling operators, into irreducible representations of the algebra. The state in each representation with the lowest scaling dimension corresponds to a primary operator.

Many models of statistical mechanics satisfy the requirement of reflection positivity (see p.6). In this case, we are interested in unitary representations of the Virasoro algebra. It turns out that, for $c < 1$, these are severely constrained, in the same way that unitarity restricts the possible representations of a finite Lie algebra. In fact, only the values of c given by

$$c = 1 - \frac{6}{m(m+1)}, \qquad \text{where } m = 3, 4, 5, \ldots \qquad (11.36)$$

are allowed, and, for each value of m, there are only a finite number of representations. The corresponding (complex) scaling dimensions of the primary operators are given by the Kac formula

$$h = h_{r,s} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)}, \qquad (11.37)$$

with $1 \leq s \leq r \leq m-1$. This result goes at least part of the way towards realising the theorist's dream, implicit in the discussion of Chapter 4, of classifying *all* fixed points, and thereby all universality classes. Among the universality classes contained in the set with $c < 1$ are some old favourites: the critical Ising model, the 'hydrogen atom' of the subject, corresponds to $m = 3$, $c = \frac{1}{2}$. The tricritical Ising model sits at $m = 4$, the three-state Potts model at $m = 5$, and so on. For these models, it turns out that all universal properties of the fixed point theory can be found analytically: not only the scaling dimensions, but the operator product expansion coefficients, the correlation functions, and more.

11.5 The c-theorem

Recent work on the subject of two-dimensional critical models has focussed less on the conformally invariant fixed point theories themselves. Rather, it has attempted to elucidate the nature and universal properties of the renormalization group flows between them. A simple result is the so-called c-theorem of A. B. Zamolodchikov. Since it is relatively simple to state and prove, yet is deep and physically compelling, it forms a suitable point at which to end this account.

The c-theorem is formulated in the continuum limit. That is, it concerns the behaviour of renormalization group flows in the subspace of all interactions where the irrelevant lattice terms which break translational and rotational symmetry have already flowed to zero. Since rotational invariance will be a crucial input, this also excludes systems which exhibit intrinsically anisotropic scaling (see p.**??**). It also assumes reflection positivity (p.6). With these provisos, the c-theorem is simply stated:

- There exists a function C of the coupling constants which is non-increasing along renormalization group flows, and is stationary only at the fixed points. Moreover, its value at each fixed point is that of the central charge c of the corresponding conformally invariant theory.

The proof relies on rotational invariance, reflection positivity, and the conservation of the stress tensor (a property which is a general consequence of translational invariance, and therefore is also valid away from the critical point). Consider some particular point on a renormalization group trajectory specified by a set of couplings $\{K\}$. For the time being, however, we suppress the dependence on $\{K\}$. Away from the fixed point, in addition to the components $T = T_{zz}$ and $\overline{T} = T_{\bar{z}\bar{z}}$ of the stress tensor, it has a non-zero trace $\Theta \equiv T^z_z + T^{\bar{z}}_{\bar{z}} = 4T_{\bar{z}z}$, since the hamiltonian is no longer invariant under dilatations. These three components, T, Θ and $\overline{T}$ have spins $s = 2, 0$ and -2 respectively under rotations. Thus their two-point functions have the form

$$\langle T(z,\bar{z})T(0,0)\rangle = F(z\bar{z})/z^4 \qquad (11.38)$$

$$\langle \Theta(z,\bar{z})T(0,0)\rangle = \langle T(z,\bar{z})\Theta(0,0)\rangle = G(z\bar{z})/z^3\bar{z} \qquad (11.39)$$

$$\langle \Theta(z,\bar{z})\Theta(0,0)\rangle = H(z\bar{z})/z^2\bar{z}^2 \qquad (11.40)$$

where F, G, and H are non-trivial scalar functions. On the other hand, conservation of the stress tensor $\partial^\mu T_{\mu z}$ implies, in complex coordinates, that

$$\partial_{\bar{z}} T + \tfrac{1}{4}\partial_z \Theta = 0. \qquad (11.41)$$

Taking the correlation function of the left hand side with $T(0,0)$ and $\Theta(0,0)$, respectively, yields two equations

$$\dot{F} + \tfrac{1}{4}(\dot{G} - 3G) = 0 \qquad (11.42)$$

$$\dot{G} - G + \tfrac{1}{4}(\dot{H} - 2H) = 0, \qquad (11.43)$$

where $\dot{F} \equiv z\bar{z}F'(z\bar{z})$, etc. On eliminating G and defining $C \equiv 2F - G - \frac{3}{8}H$, these reduce to

$$\dot{C} = -\tfrac{3}{4}H. \qquad (11.44)$$

Now reflection positivity requires that $\langle\Theta\Theta\rangle \geq 0$, so that $H \geq 0$. Thus C is a non-increasing function of $R \equiv (z\bar{z})^{1/2}$, and is stationary only when $H = 0$.

Now imagine making a renormalization group transformation $a \to a(1+\delta\ell)$. Since $C(R,\{K\})$ is dimensionless, this is equivalent to sending $R \to R(1-\delta\ell)$, and the coupling constants $\{K\}$ will flow according to the renormalization group equations. Thus

$$\left(\frac{d}{d\ell} - R\frac{\partial}{\partial R}\right) C(R,\{K\}) = 0. \qquad (11.45)$$

If we now define $C(\{K\}) \equiv C(R_0,\{K\})$, where R_0 is some arbitrary but fixed length scale, we see that this quantity satisfies the first part of the c-theorem. Moreover, it is stationary if and only if $H = 0$, which, by reflection positivity, implies $\Theta = 0$, so that the theory is scale invariant and therefore corresponds to a fixed point. Finally, at such a fixed point, $G = H = 0$, and $F = \frac{1}{2}c$, so that indeed $C = c$.

The c-theorem has the interpretation that renormalization group flows go 'downhill'. In particular, it rules out the existence (for systems satisfying reflection positivity) of limit cycles and other esoteric behaviour in renormalization group flows. It also severely restricts the possible fixed points to which unstable directions at a given fixed point may flow. For example, relevant operators at the tricritical Ising fixed point, corresponding to $m = 4$ in the classification (12.36), may generate flows into either trivial fixed points with $c = 0$ (for example, high or low temperature fixed points), or to fixed points in the universality class of the critical Ising model, with $m = 3$, $c = \frac{1}{2}$. For this reason, the critical behaviour on the edges of the wings in the tricritical phase diagram shown in Figure **??** must be in the critical Ising universality class, despite the lack of any obvious symmetry.

An appealing physical interpretation of the C-function is as a kind of entropy of information about the critical system. Under renormalization, information is lost about the short distance behaviour of the correlation functions. However, this cannot be taken too literally – for example, even at infinite temperature a block spin transformation results in loss of information about the microstates of the system, yet no renormalization group flow takes place. Presumably, a more complete interpretation along these lines needs to account for the fact that the central charge is sensitive to only the effectively gapless degrees of freedom. Such a picture, if validated, would presumably extend to higher dimensions. However, so far, all attempts to prove higher-dimensional versions of the full c-theorem have failed. It is not difficult to satisfy the requirements of the theorem locally. The problem is to find a suitable function which is globally defined, is finite at each fixed point and is, at least in principle, measurable solely in terms of the correlation functions there.

Exercises

11.1 Show that the inversion transformation described in Section 12.1 is a conformal transformation, in any number of dimensions.

11.2 In the half space $z > 0$, when the order parameter is fixed to some non-zero value on the plane $z = 0$, its expectation value at the bulk critical point decays as z^{-x}, where x is its scaling dimension. By making an inversion about a suitable origin, find its behaviour in the interior of a sphere of radius R, with fixed boundary conditions on $r = R$.

11.3 By conformally mapping the upper half plane (with fixed boundary conditions on the order parameter) into a strip of width L, show that the correlation function along the strip decays exponentially at large distances, with a correlation length $\pi x^{(s)}/L$, where $x^{(s)}$ is the boundary scaling dimension of the order parameter (see Section **??**).

11.4 By conformally mapping the upper half plane into a wedge of opening angle θ, show that an operator near the apex of the wedge has a scaling dimension $(\theta/\pi)x^{(s)}$, where $x^{(s)}$ is its boundary scaling dimension. Using scaling arguments, show that, below the bulk critical temperature the order parameter near the apex vanishes as $(-t)^{\beta(\theta)}$, and determine the dependence of this exponent on the opening angle.

11.5 The Gaussian model corresponding to the line of low-temperature fixed points of the two-dimensional XY model (Section **??**) is a simple example of a conformally invariant system. Using the rules of Gaussian integration described in the
Appendix, calculate the three-point function $\langle e^{iq_1\theta(r_1)}e^{iq_2\theta(r_2)}$ $e^{iq_3\theta(r_3)}\rangle$ and show that it has the form given in (12.13). [Note that this correlation function vanishes unless $\sum_i q_i = 0$.]

11.6 Show that the specific heat of a one-dimensional gas of massless relativistic bosons has the form (12.34), and check that $c = 1$. Repeat the calculation for particles obeying Fermi–Dirac statistics. What is the value of c in this case?