Notes on SLE

These notes are an addendum to my 2008 les Houches lecture notes, where some familiarity with SLE was assumed. For further details, see, eg, http://www-thphys.physics.ox.ac.uk/people/JohnCardy/slereview.pdf

Schramm-Loewner evolution is a probabilistic theory of the scaling limit of the planar curves which arise in 2d lattice models, e.g. those in the ADE and $O(n)$ models we have already met.

There are two different lines of argument: the first assumed a couple of simple axioms about the conformal invariance of this assumed scaling limit. From these many rigorous results may be derived as well as the relation to (boundary) CFT. The second, which is harder, actually proves that these axioms hold for the scaling limit of particular lattice models. So far this has been carried out in only a few cases.

Consider a simple curve $\gamma$ in the upper half plane $\mathbb{H}$ from the point $a_0$ on the real axis to $\infty$. In mathematical terms such a curve comes along with a continuous parametrisation $0 \leq t < \infty$. Later we will choose a particular one. Let $\gamma_t$ be the part of the curve up to 'time' $t$. Then $\mathbb{H} \setminus \gamma_t$ is a simply connected region, which, by the Riemann mapping theorem, may be conformally mapped back to $\mathbb{H}$ by $z \mapsto g_t(z)$. We can think of $\mathbb{H}$ as being opened up by a slit along $\gamma_t$. The L and R sides of this slit are mapped to the real axis. The function $g_t(z)$ is not unique but we can make it so by demanding that

$$g_t(z) = z + O(1/z)$$

as $z \to \infty$ (that is, no constant term.) It can also be shown that as the curve grows the coefficient of the $O(1/z)$ term always increases, so we can define $t$ so that $g_t(z) = z + 2t/z + O(1/z^2)$. In that case the growing tip gets mapped to a point $a_t$ on the real axis. $a_t$ is a continuous function of $t$.

The simplest example is when $\gamma$ is a straight line perpendicular to the real axis. In that case

$$g_t(z) = (z - a_0)^2 + 4t)^{1/2} + a_0$$

Even though we expect the curve in a critical model to be fractal and therefore non-differentiable, the map $g_t(z)$ should evolve smoothly as long as $z$ is not on the curve. We can get an equation for this by noting that

$$g_{t+\delta t}(z) \approx ((g_t(z) - a_t)^2 + 4\delta t)^{1/2} + a_t$$

or

$$dg_t(z) = \frac{2dt}{g_t(z) - a_t}$$

This is (ordinary) Löwner evolution. It shows that the study of simple curves $\gamma_t$ growing into $\mathbb{H}$ is equivalent to looking at continuous functions $a_t$. If $a_t$ is differentiable the curve
is smooth and we may write as an ordinary differential equation point-wise in $z$. Given $\gamma_t$ we may infer $a_t$ and vice versa. In our case this is a stochastic differential equation, and it is not obvious for which functions $a_t$ we will get a continuous curve (but it is OK for SLE).

In SLE we are concerned with defining a measure on curves $\gamma$ and hence a measure on functions $a_t$. This may be determined by assuming two basic axioms:

1. Domain Markov property: the measure on the rest of the curve $\gamma \setminus \gamma_t$, given $\gamma_t$, in $\mathbf{H}$, is the same as the measure on curves from the tip of the slit to $\infty$ in the domain $\mathbf{H} \setminus \gamma_t$. This is obviously true, even on the lattice, for a domain wall in the ADE models with neighbouring fixed heights on the negative and positive real axes.

2. Conformal Invariance: if we map $\mathbf{H} \setminus \gamma_t$ back to $\mathbf{H}$ using $g_t$, the induced measure on $g_t(\gamma \setminus \gamma_t)$ is the same as that on the original $\gamma$ (shifted by $a_t - a_0$).

Then a theorem due to Schramm states that the only possibility for $a_t$ is a Brownian motion starting at $a_0$. The only free parameter is the diffusion constant:

$$\langle a_t - a_0 \rangle = 0, \quad \langle (a_t - a_{t'})^2 \rangle = \kappa |t - t'|$$

The proof goes by using (1,2) to show that $a_t$ must obey a law of independent increments, and have zero drift. As we will see shortly, $\kappa$ is related to the Coulomb gas coupling by $g = 4/\kappa$, and hence to the central charge $c$. The dilute critical point corresponds to $\kappa \leq 4$, and for this it has been proved that $\gamma$ is indeed a simple curve. In the dense phase $\gamma$ continually self-intersects (but does not cross itself). In that case the original Loewner argument still applies, replacing $\gamma_t$ by its ‘hull’ $K_t$, the region enclosed between $\gamma_t$ and the real axis.