Solutions 1

1. Consider an inversion $r \rightarrow r/|r|^2$. The Jacobian is $|r|^{-2}$, so

$$|r_1 - r_2|^{-x_1 - x_2} = |r_1|^{-2x_1}|r_2|^{-2x_2} \left| \frac{r_1}{|r_1|^2} - \frac{r_2}{|r_2|^2} \right|^{-x_1 - x_2}$$

This is an identity if $x_1 = x_2$ but cannot be true otherwise: for example by taking $r_1 \rightarrow 0$ we see that it is false. Note that this uses only the special conformal group so is true for primary fields in higher dimensions.

2. Choose spherical polar coordinates $(R, \theta, \phi)$ on the sphere, with metric $ds^2 = R^2(d\theta^2 + \sin^2 \theta d\phi^2)$. The plane is tangent to the sphere at the N pole $\theta = 0$, and the projection $P'$ of a point $P$ is where a line from the S pole through $P$ intersects this plane. This gives $P'$ to be at $(\rho, \phi)$ in polar coordinates, where $\rho = 2R \sin \frac{\theta}{2}$. This is conformal because $ds^2 = \cos^2 \theta (d\rho^2 + \rho^2 d\phi^2)$. Hence

$$\langle \Phi(\theta_1, \phi_1)\Phi(\theta_2, \phi_2) \rangle_{S^2} = \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right]^x \left[ 4R^2 \sin^2 \frac{\theta_1}{2} + 4R^2 \sin^2 \frac{\theta_2}{2} - 8R^2 \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \cos(\phi_1 - \phi_2) \right]^x = R^{-2x} \left[ \sin^2 \theta_1 + \sin^2 \theta_2 - 2 \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) \right]^{-x}$$

Note that the expression in brackets is the (square of) the chordal distance between the two points on the sphere.

3. $$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2) \rangle_{\text{cyl}} = \left| \frac{dz_1}{dw_1} \right|^x \left| \frac{dz_2}{dw_2} \right|^x \frac{1}{|z_1 - z_2|^{2x}}$$

After some algebra this gives, with $w = r + i\tau$,

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2) \rangle_{\text{cyl}} = \left( \frac{2\pi}{\beta} \right)^{2x} \left[ 2 \cosh \left( \frac{2\pi(r_1 - r_2)}{\beta} \right) - 2 \cosh \left( \frac{2\pi(t_1 - t_2)}{\beta} \right) \right]^{2x}$$

Note this decays exponentially as $|r_1 - r_2| \rightarrow \infty$, with the thermal correlation length $\xi \sim 2\pi x/\beta$. Continuing to real time we find (setting $r_2 = t_2 = 0$ for convenience)

$$\langle \phi(r, t)\phi(0, 0) \rangle_{\beta} = \frac{(2\pi/\beta)^{2x}}{\left[ 2 \cosh \left( \frac{2\pi r}{\beta} \right) - 2 \cosh \left( \frac{2\pi t}{\beta} \right) \right]^{2x}}$$

However, this is valid only outside the light cone $t < |r|$. A naive continuation beyond this suggests that it gains an imaginary part $\propto \sin(\pi x)$. In some lattice models this agrees with explicit results, taking first $t > |r|$ and then the continuum limit.
4. (a) use the conformal mapping

\[ w = iR \left( \frac{z - i}{z + i} \right) \]

which takes \( \text{Im} \, z > 0 \) into \( |w| < R \). Hence

\[ \langle \phi(w, \bar{w}) \rangle_{\text{disc}} = \left| \frac{dz}{dw} \right|^x \frac{1}{(\text{Im} \, z)^x} = \left( \frac{2R}{R^2 - |w|^2} \right)^x \]

(b) use \( w = (L/\pi) \log z \). The same sort of argument gives

\[ \langle \phi(w = iy) \rangle_{\text{strip}} = \left( \frac{\pi/L}{\sin(\pi y/L)} \right)^x \]

5. The normalisation constant in front of the action is somewhat arbitrary. We can choose it so the 2-point functions are

\[ \langle \psi(z_1) \psi(z_2) \rangle = \frac{1}{z_1 - z_2}, \quad \langle \bar{\psi}(\bar{z}_1) \bar{\psi}(\bar{z}_2) \rangle = \frac{1}{\bar{z}_1 - \bar{z}_2} \]

The stress tensor \( T \) may be found using Noether’s theorem, of simply by observing that \( T = \alpha \psi \partial_z \psi \) is the only bilinear in the fields which has the correct scaling dimensions \((2, 0)\). The constant \( \alpha \) may be fixing by demanding the \( T \) satisfy the correct OPE with \( \psi \). By Wick’s theorem (note that the contraction between the \( \psi(z) \) at the same point is removed by point-splitting and subtracting this contribution)

\[
\psi(z) \partial_z \psi(z) \cdot \psi(z_1) = \psi(z) \partial_z \left( \frac{1}{z - z_1} \right) - \partial_z \psi(z) \frac{1}{z - z_1} + \cdots \\
= -\frac{1}{(z - z_1)^2} \psi(z_1) + (z - z_1) \partial_z \psi(z_1) - \frac{1}{z - z_1} \partial_z \psi(z_1) + \cdots \\
= -\frac{1}{(z - z_1)^2} \psi(z_1) - \frac{2}{z - z_1} \partial_z \psi(z_1) + \cdots
\]

so we should take \( \alpha = -\frac{1}{2} \) to get the last term right. This also gives \( \Delta_\psi = \frac{1}{2} \) as expected. The 2-point function is now

\[
\langle T(z_1)T(z_2) \rangle = \frac{1}{2^2} \langle \psi(z_1) \partial_{z_1} \psi(z_1) \psi(z_2) \partial_{z_2} \psi(z_2) \rangle \\
= \frac{1}{4} \left[ -\frac{1}{z_1 - z_2} \partial_{z_1} \partial_{z_2} \frac{1}{z_1 - z_2} + \partial_{z_1} \frac{1}{z_1 - z_2} \partial_{z_2} \frac{1}{z_1 - z_2} \right] \\
= \frac{1}{4} \frac{2 - 1}{(z - 1 - z_2)^4} \quad \text{so that } c = \frac{1}{2}
\]