Fields Medal 2010 for Stanislav Smirnov, University of Geneva

The Community of Swiss Physicists is very happy and honoured that Stanislav Smirnov, professor in the Analysis, Mathematical Physics and Probability group of the University of Geneva since 2003, has won the Fields Medal 2010, the most prestigious prize in mathematics. He was rewarded for his proof of conformal invariance at criticality both for site percolation on the triangular lattice and for the two-dimensional Ising model. Since he is close to physics not only in spirit but also in action, as for example by organizing the "Physical Mathematics Seminars" in Geneva, it is very natural to acknowledge his achievement in our journal. We are grateful to Prof. John Cardy for retracing the extraordinary story leading from the early days of critical phenomena to this mathematical tour de force. John Cardy has himself played a central role in the formulation of the problem, specifically through a conjecture, now widely known as the Cardy formula, for certain correlations in critical percolation. His article shows nicely how insights from both physicists and mathematicians are intertwined in the exciting achievement of Stanislav Smirnov.

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Conformal Invariance and the Scaling Limits of Lattice Models

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At the International Congress of Mathematicians, held in Hyderabad, India, Stanislav Smirnov of the University of Geneva was awarded the 2010 Fields Medal for the proof of conformal invariance of percolation and the planar Ising model in statistical physics. As most people know, the Fields Medal is regarded as the mathematics equivalent of a Nobel Prize (although it is awarded only to those not over the age of 40.)

What was Smirnov's achievement, and why is it important for physics? His work can be seen as the rigorous mathematical treatment of a set of ideas originating in theoretical physics in the late 1960s. These concerned the behaviour of systems close to a critical point, or second order phase transition. A famous example is the Ising model of ferromagnetism, first formulated by Lenz. In this model quantum spins, which can be 'up' or 'down', are situated at the sites of a regular lattice. There is an interaction between them which favours neighbouring spins being in the same state, either both up or both down. At low temperatures the system is a ferromagnet: a majority of spins are all up, or all down. At high temperatures there is no net magnetisation. These two regimes are separated by a critical point. Although certain properties of the square lattice Ising model were famously computed exactly by Onsager much earlier, a more physical understanding came with the realisation by Wilson, Fisher, Kadanoff and others that at the critical point the system is scale invariant. This means that if we were to take a photograph of such a system at the critical point (colouring, for example, up spins black and down spins white), and then blow up the picture by some factor $b>1$, then (ignoring the detailed graininess due to the lattice) the pictures would look statistically similar – we could not differentiate between the original or the blown-up version.

This idea was at the basis of the renormalisation group, leading to a deep understanding of the different kinds of critical behaviour which can occur in nature. For this work Wilson was awarded the Nobel Prize in 1982. However not long after the first papers, in 1970, the young Russian physicist Alexander Polyakov [1] generalised the idea of scale invariance to conformal invariance: he argued that the photograph should also remain statistically similar when the blow-up factor b is allowed to vary smoothly as a function of position. These transformations mathematically are called *conformal mappings*: they locally preserve angles but not volumes. In the 1980s a whole edifice of theoretical physics was erected on the basis of Polyakov's hypothesis: it led to a host of new analytic results in critical behaviour, especially in two dimensions, and went under the name of conformal field theory (CFT). Interestingly, the mathematical tools required for this were first developed in string theory. They imply that the correlation functions of local observables (for example the spins in the Ising model) obey certain linear differential equations.

However, there was no proof that the scaling limit (where the grains of Fig. 1 are supposed to be infinitesimally small) is in fact conformally invariant, for the Ising model or any other similar system. The situation came to a head when theoretical physicists began to apply CFT to percolation. This is an even simpler problem than the Ising model: the sites of the lattice are independently labelled either white (with probability p) or yellow (with probability 1– p). The interest comes in asking questions about clusters of white or yellow spins. For small p there are only small clusters of white spins in a sea of yellow, and vice versa if p is close to one. Suppose the whole system is contained in a large rectangle. There exists a critical value p_c at which a white cluster (say) is just able to span the system from top to bottom. At $p=p_c$, what is the probability of this happening? The answer depends on the aspect ratio of the rectangle: a tall narrow one is more difficult to cross than a short fat one. In 1992 an exact formula was conjectured [2] using ideas of CFT applied to percolation.

The key point was to regard the places where the boundary conditions change (in this example the corners of the rectangle) as local observables similar to the spins themselves, so that the crossing probability obeys a similar sort of linear differential equation in CFT. The solution of this, known as the 'Cardy formula', is quite non-trivial, involving hypergeometric functions, but it agrees very well with the results of numerical simulations and seemed to be correct. However its existence presented a puzzle to probabilists who had been working on percolation for many years, as it seemed to be far beyond the reach of their existing methods, and the CFT approach rested on far too many unproven hypotheses.

In 2000 a breakthrough came in the form of a new description of conformally invariant curves, called Schramm-Loewner evolution (SLE), developed by Oded Schramm [3] and his coworkers Greg Lawler and Wendelin Werner (who also received a Fields Medal, in 2006). This is too complicated to explain here, but it laid out a new mathematical framework within which to discuss these problems. And Schramm showed that if the curves which form the cluster boundaries of percolation clusters satisfy the postulates of SLE, Cardy's formula would follow.

Enter Smirnov. In 2001 he actually proved [4] the Cardy formula from first principles, for a particular lattice percolation model. The essentials of his argument are relatively simple: he showed that a certain local observable of percolation is discretely holomorphic: that is, it is approximately given by the real part of an analytic function. Moreover the approximation gets better and better as the lattice spacing approaches zero. Complex analytic functions are well-known to be linked to conformal mappings in two dimensions: as undergraduates we learn how to use them to solve problems in electrostatics. Once one knows the behaviour of such a function on the boundary, one knows it everywhere. This line of argument led Smirnov to his proof of the Cardy formula. Moreover, he showed it to be true in an arbitrary region, and that, it turns out, is the main requirement to reversing Schramm's argument and showing that the boundaries of percolation clusters are indeed described by SLE.

This was not his only achievement cited for the Fields Medal. In 2006 he announced a similar result [5] for curves in the two-dimensional Ising model, giving at last a firm mathematical basis for Polyakov's 1970 hypothesis.

These mathematical results are not simply a matter of dotting 'i's and crossing 't's in arguments which were already well known to theoretical physicists. SLE has led to a whole new way of understanding certain aspects of CFT. Smirnov's discrete holomorphicity has been verified for many other two-dimensional lattice models (although the proof of convergence to an analytic function in these other cases is still missing) and there also appears to be a (so far) poorly understood relationship to integrability, a completely different field of mathematical physics but one which underlies Onsager's original results. This striking example of cross-fertilisation between physics and mathematics is likely to have a long way yet to run.

[1] A. M. Polyakov, Zh. Eksp. Teor. Fiz. **12**, 538 (1970) [Sov. Phys. JETP Lett. **12**, 381 (1970)].
[2] J. Cardy, J. Phys. A **25**, L201 (1992).
[3] O. Schramm, Israel J. Math. **118**, 221 (2000).
[4] S. Smirnov, C. R. Acad. Sci. Paris Sér. I Math., **333**, no. 3, 239 (2001).
[5] S. Smirnov, in Sanz-Solé, Marta (ed.) et al., Proceedings of the International Congress of Mathematicians (ICM), Madrid, Spain, August 22-30, 2006. Volume II, 1421--1451. Zürich: European Mathematical Society, 2006.

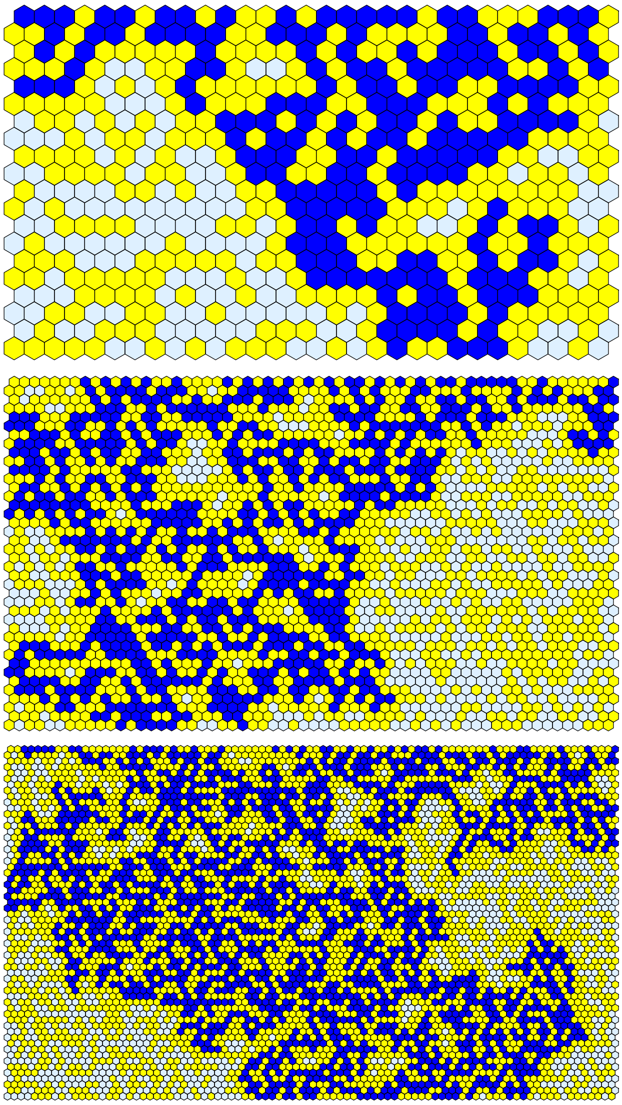

Fig. 1 (courtesy S. Smirnov): Percolation of water through a 'rock', on different scales. The rock is represented by yellow, and the water, shown in blue, can run from top to bottom only through the spaces in between. The spaces not reached by the water are shown in white. The water occupies a cluster which spans the system from the top edge to the bottom edge. Cardy's formula, proved by Smirnov, gives the probability that such a cluster exists, in the scaling limit when the size of the pores gets infinitely fine. SLE gives information about the fractal curve which forms the boundary of this cluster.