Statistical Mechanics: Set 2

1. The states of a system have energy E_α, angular momentum J_α and are occupied with probability p_α. Show that the system's entropy is maximized subject to specified $\langle E\rangle$ and $\langle J\rangle$ by the distribution

$$p_\alpha = \frac{1}{Z}\,\mathrm{e}^{-\beta(E_\alpha - \Omega J_\alpha)},$$

where β and $\lambda \equiv \beta\Omega$ are Lagrange multipliers and Z is the partition function. Give expressions for $\langle E\rangle$ and $\langle J\rangle$ in terms of β, Ω, and the Helmholtz free energy F.

Show that the mean-square fluctuation in the system's angular momentum is

$$\left\langle(\Delta J)^2\right\rangle = \frac{1}{\beta}\frac{\partial\langle J\rangle}{\partial\Omega}.$$

How does the rms fluctuation in J scale with the size of the system?

2. The dispersion relation for surface-tension ripples on a liquid is $\omega^3 = \Gamma k^3/\rho$, where ρ is the liquid's density and Γ is its surface tension. Show that the free energy of ripples per surface atom is $-k_{\rm B}T(T/T_0)^{4/3}$, where T_0 is a constant with the dimensions of temperature.

Deduce the corresponding entropy per surface atom.

Given that for He, $\rho = 150\,\mathrm{kg\,m^{-3}}$, $\Gamma = 3.1\times10^{-4}\,\mathrm{N\,m^{-1}}$ and the area occupied by one atom is $1.6\times10^{-19}\,\mathrm{m}^2$, calculate T_0 for He. $\left[\int_0^\infty \mathrm{d}x\, x^{4/3}/(\mathrm{e}^x-1) = 1.68\right]$

3. Consider an array of N_a non-interacting atoms. Each atom has a single-particle state that can be (a) unoccupied, (b) occupied by a single electron of energy E_0, (c) occupied by two electrons with oppositely directed spins and total energy $2E_0+U$. Denoting the magnetic dipole moment of an electron by m, write down an expression for the grand partition function of the array in magnetic field B. Hence obtain the thermodynamic potential $\Phi(\mu, N_a, T, B)$.

Find (i) the average number of electrons in the array; (ii) the magnetic susceptibility at $B=0$; (iii) the specific heat capacity at $B=0$.

4. Model an atomic nucleus by two coextensive zero-temperature Fermi gases in a volume $\frac{4}{3}\pi r_0^3 A$, where $r_0 \sim 1.2\times10^{-15}\,\mathrm{m}$ is an effective nucleon radius and A is the nucleon number $A = Z+N$. Here Z the number of protons and N the number of neutrons in the nucleus. Calculate the Fermi energies of the proton and neutron components. Suppose that protons could reversibly change into neutrons, and vice versa. For A even, show that minimum-energy configuration of the nucleus would then be that in which $Z = N = A/2$. Show that for $Z - A/2$ small, the energy is

$$E(Z) = E(A/2) + C(Z - \tfrac{1}{2}A)^2/A,$$

and estimate the constant C for a nucleus of mass $2\times10^{-27}\,\mathrm{kg}$. Estimate the temperature at which the assumption $T=0$ yields significant error.

5. A simple model of a ferromagnet is defined as follows. N spins are represented by the variables $\sigma_i = \pm 1$. The spins interact with each other and with an external field b, and the system's Hamiltonian is

$$H = -\frac{J}{N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - b\sum_{i=1}^{N}\sigma_i.$$

Show that the thermally averaged magnetization $\langle\sigma_i\rangle$ and the free energy per spin, f, are related by

$$\langle\sigma_i\rangle = -\frac{\partial f}{\partial b}.$$

Show that H is a function of $S \equiv \sum_i \sigma_i$. What values may S take? Give an expression for the number of states of the magnet associated with each given value of S and write the partition function as a sum over the allowed values of S.

For $N \gg 1$ approximate the sum by its largest term. Show that in this approximation f is the maximum over x of

$$f(x) = k_\mathrm{B}T\left[\tfrac{1}{2}(1+x)\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x) - \ln 2\right] - Jx^2 - bx.$$

Show that when $b = 0$ the model has a phase transition at temperature $T_c = 2J/k_\mathrm{B}$.

6. The Hamiltonian of an Ising antiferromagnet differs from that of an Ising ferromagnet only in a change in the sign of the coefficient $\mathcal{J}$ of the spin-spin interaction. Show that if the lattice is square, there is a ferromagnetic system whose states can be put into one-to-one correspondence with those of the antiferromagnet, such that corresponding states have the same energy.

7. A one-dimensional chain of A and B atoms has nearest-neighbour interaction energies ϵ_{AA}, ϵ_{AB} and ϵ_{BB} according as the neighbours are both A atoms, etc. Let $S_i = 1$ if site i is occupied by an A atom, and $S_i = -1$ if it is occupied by a B atom. Show that the Hamiltonian is then of the form

$$H = -J\sum_i S_i S_{i+1} + B\sum_i S_i + C$$

and obtain expressions for J, B and C in terms of ϵ_{AA} etc.

Show that the free energy per spin, F, in the case $B = C = 0$ is

$$\beta F = -\beta J - \ln\left(1 + \mathrm{e}^{-2\beta J}\right),$$

where β is the usual inverse temperature. Hence obtain an expression for the internal energy per spin and discuss its behaviour in the limits $T \to 0$ and $T \to \infty$.