Solar-System Dynamics

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1 Orbital elements

- Solar system very old: 5,000,000,000 rotations of earth around Sun; 50,000,000,000 rotations of Moon around Earth.
- Planetary trajectories almost solved by Kepler in $17^{\rm th}$ century; interest now focuses on *slow* evolution of orbits from one Kepler ellipse to another.
- Kepler ellipses valid so long as can consider just two bodies: Earth and Sun, or Moon and Earth. Slow evolution driven by third, fourth,. . . , bodies. Evolution slow because 2-body approx good: e.g. $\mathrm{M}_\odot/M_J = 1047$, $a_J/a_E = 5.2$, so force exerted by Jupiter on Earth smaller by factor $\gtrsim 10^4$ than that exerted by Sun.
- The problem perfectly suited to perturbation theory, with small parameter $M_J/\mathrm{M}_\odot$ etc:

Step 1: Solve unperturbed (Kepler) problem.

Step 2: Describe general motion in terms of instantaneous Kepler orbit. 'Orbital elements' specify the current Kepler orbit ('osculating ellipse').

Step 3: Obtain and solve equations of motion of orbital elements.

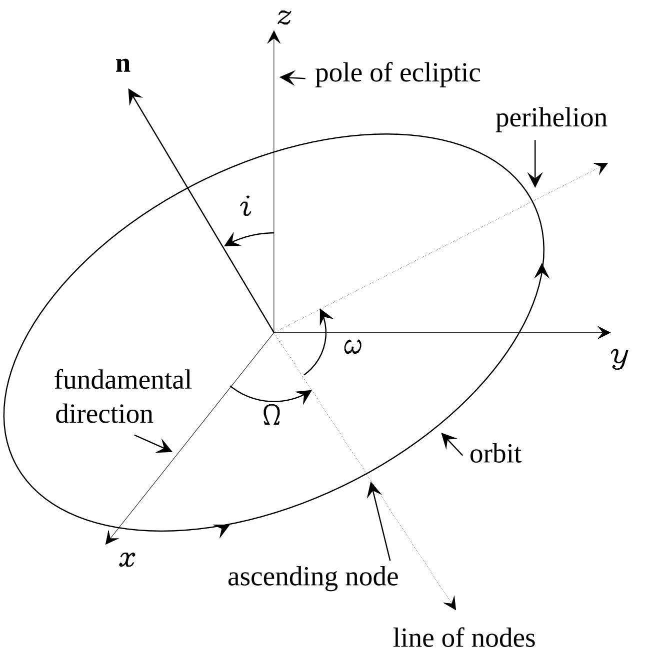
Choose a plane, the **fundamental plane**, fixed in inertial space (current ecliptic) and a **fundamental direction** (the Earth-Sun line at the current vernal equinox). Then any Kepler orbit about the Sun is confined to a plane inclined with respect to this fundamental plane at an angle i called the **inclination**. The orbital and fundamental planes intersect in a line called the **line of nodes**. The fundamental line and the line of nodes make an angle Ω called the **argument of the ascending node**. The orbit is an ellipse of **semi-major axis** $a = GM/2|E|$ and **eccentricity** $e \equiv \sqrt{1 - b^2/a^2} = \sqrt{1 - L^2/GMa}$. The line of nodes and the Sun-pericentre line make an angle ω, the **argument of perihelion**. These 5 **orbital elements** define a Kepler ellipse.

The orbital elements of satellites are defined analogously by substituting a planet for the Sun.

Exercise (1):

By considering a Kepler orbit in plane polar coordinates (r, ϕ), show that for such an orbit $r^{-1} = C\cos(\phi - \phi_0) + GM/L^2$. Hence show that $a = GM/2|E|$ and $e = \sqrt{1 - L^2/GMa}$.

Since the solar system isn't a two-body system, the orbital elements of a planet or satellite are all (slowly-varying) functions of time. An elegant derivation of the equations that govern their motion requires a review of Hamiltonian mechanics.



2 Review of Hamiltonian Mechanics

2.1 Poisson brackets and symplectic structure

Newton's laws are valid only in inertial Cartesian coordinates, $(\mathbf{x}, \mathbf{v})$. In terms of these coordinates one defines the **Poisson bracket** $[A, B]$ of any two functions $A(\mathbf{x}, \mathbf{v})$, $B(\mathbf{x}, \mathbf{v})$ on phase space by

$$[A, B] \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{v}} - \frac{\partial A}{\partial \mathbf{v}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{2.1}$$

It is straightforward to verify that the coordinates $(\mathbf{x}, \mathbf{v})$ satisfy the **canonical commutation relations**

$$[v_i, v_j] = [x_i, x_j] = 0 \quad \text{and} \quad [x_i, v_j] = \delta_{ij}. \tag{2.2}$$

If we write $(z_i \equiv x_i,\ z_{3+i} \equiv v_i \quad i = 1, 2, 3)$, and define the **symplectic matrix c** by

$$c_{\alpha\beta} \equiv [z_\alpha, z_\beta] = \begin{cases} \pm 1 & \text{for } \alpha = \beta \mp 3,\ 1 \le \alpha, \beta \le 6; \\ 0 & \text{otherwise,} \end{cases} \tag{3a}$$

we have

$$[A, B] = \sum_{\alpha,\beta=1}^{6} c_{\alpha\beta} \frac{\partial A}{\partial z_\alpha} \frac{\partial B}{\partial z_\beta}. \tag{2.3b}$$

Any set of 6 phase-space coordinates $\{Z_\alpha,\ \alpha = 1,\ldots,6\}$ is called a set of **canonical coordinates** if $[Z_\alpha, Z_\beta] = c_{\alpha\beta}$. Let $\{Z_\alpha\}$ be such a set; then with equation (3b) and the chain rule we have

$$\begin{aligned}[A,B] &= \sum_{\alpha,\beta=1}^{6} c_{\alpha\beta}\frac{\partial A}{\partial z_\alpha}\frac{\partial B}{\partial z_\beta} = \sum_{\kappa\lambda}\left(\sum_{\alpha\beta} c_{\alpha\beta}\frac{\partial Z_\kappa}{\partial z_\alpha}\frac{\partial Z_\lambda}{\partial z_\beta}\right)\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda} \\ &= \sum_{\kappa\lambda}[Z_\kappa, Z_\lambda]\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda} = \sum_{\kappa\lambda} c_{\kappa\lambda}\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda}.\end{aligned} \tag{2.4}$$

Thus the derivatives involved in the definition (2.1) of the Poisson bracket can be taken with respect to any set of canonical coordinates, just as the vector formula $\nabla\cdot\mathbf{a} = \sum_i(\partial a_i/\partial x_i)$ is valid in any Cartesian coordinate system. It is conventional to denote the first 3 coordinates W_i by q_i and the last 3 by $p_i = W_{3+i}$.

The **Hamiltonian** is just the particle's energy written as a function of the phase-space variables. For Kepler motion it is

$$H_K(\mathbf{x},\mathbf{v}) = \tfrac{1}{2}v^2 - \frac{GM}{r}. \tag{2.5}$$

Hamilton's equation express the rate of change of $(\mathbf{x},\mathbf{v})$ in terms of gradients of H:

$$\dot{x}_i = \frac{\partial H}{\partial v_i} \quad ; \quad \dot{v}_i = -\frac{\partial H}{\partial x_i}. \tag{2.6}$$

Hamilton's eqns may be written

$$\dot{x}_i = [x_i, H] \quad ; \quad \dot{v}_i = [v_i, H]. \tag{2.7}$$

The rate of change of an arbitrary canonical coordinate Z_α along an orbit is

$$\dot{Z}_\alpha = \sum_{\beta=1}^{6}\frac{\partial Z_\alpha}{\partial z_\beta}\dot{z}_\beta, \tag{2.8}$$

where, as usual, $\mathbf{z} \equiv (\mathbf{x},\mathbf{v})$. With Hamilton's equations (2.7) and equation (2.4) this becomes

$$\begin{aligned}\dot{Z}_\alpha &= \sum_{\beta=1}^{6}\frac{\partial Z_\alpha}{\partial z_\beta}[z_\beta, H] = \sum_{\beta\gamma\delta}\frac{\partial Z_\alpha}{\partial z_\beta}c_{\gamma\delta}\frac{\partial z_\beta}{\partial z_\gamma}\frac{\partial H}{\partial z_\delta} = \sum_{\gamma\delta}c_{\gamma\delta}\frac{\partial Z_\alpha}{\partial z_\gamma}\frac{\partial H}{\partial z_\delta} \\ &= [Z_\alpha, H].\end{aligned} \tag{2.9}$$

Thus Hamilton's equations (2.7) are valid in *any* canonical coordinate system.

A neat way of making canonical transformations is to use a **generating function**. For example, suppose you have a function $S(\mathbf{P},\mathbf{q})$ of some new variables P_i, $i = 1,2,3$ and the 'coordinates' q_i of a canonical system such that the equation

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \tag{2.10a}$$

can be interpreted as defining $\mathbf{P}(\mathbf{p},\mathbf{q})$. Then a straightforward calculation (see Appendix I) shows that the coordinates $(\mathbf{P},\mathbf{Q})$ are canonical, where

$$\mathbf{Q} \equiv \frac{\partial S}{\partial \mathbf{P}}. \tag{2.10b}$$

That is, $[Q_i, Q_j] = 0$, $[Q_i, P_j] = \delta_{ij}$, $[P_i, P_j] = 0$. The transformation $(\mathbf{p},\mathbf{q}) \to (\mathbf{P},\mathbf{Q})$ is called a **canonical transformation** and S the **generating function** of the transformation.

3 Hamilton-Jacobi Equation

A **constant of motion** is a function $I(\mathbf{q},\mathbf{p})$ on phase space which takes the same value at all points on an orbit:

$$0 = \frac{\mathrm{d}I}{\mathrm{d}t} = \sum_\alpha \frac{\partial I}{\partial z_\alpha}\dot{z}_\alpha. \tag{3.1}$$

Suppose we could find 3 constants of motion I_1, I_2, I_3. And suppose it were possible to find a system of canonical coordinates $(\mathbf{J},\mathbf{w})$ such that $J_i = I_i$ etc. Then the equations of motion for the J's would be trivial,

$$\begin{aligned} 0 = \dot{J}_i &= [J_i, H] \\ &= -\frac{\partial H}{\partial w_i}. \end{aligned} \tag{3.2}$$

and would demonstrate that $H(\mathbf{J})$ would be independent of the w's. This last observation would allow us to solve the equations of motion for the w's: we would have

$$\dot{w}_i = \frac{\partial H}{\partial J_i} \equiv \omega_i(\mathbf{J}), \quad \text{a constant} \quad \Rightarrow \quad w_i(t) = w_i(0) + \omega_i t. \tag{3.3}$$

So everything would lie at our feet if we could find 3 constants of the motion and could embed these as the 'momenta' of a system of canonical coordinates.[1] The magic coordinates $(\mathbf{J},\mathbf{w})$ are called **action-angle** coordinates, the J's being the actions and the w's the angles. If the orbit is bound, the Cartesian coordinates x_i cannot increase without limit as the w_i do. From this we infer that the x_i are periodic functions of the w_i. In other words if we scale the w_i correctly, we can expand $\mathbf{x}$ in a Fourier series

$$\mathbf{x}(\mathbf{w},\mathbf{J}) = \sum_{\mathbf{n}} \mathbf{X}_{\mathbf{n}}(\mathbf{J}) e^{\mathrm{i}\mathbf{n}\cdot\mathbf{w}}, \tag{3.4}$$

where the sum is over all vectors $\mathbf{n}$ with integer components, and the w_i have been scaled so that $\mathbf{x}$ returns to its original value after w_i has increased by 2π. The 3-surface

[1] Notice that to be able to embed the J's as a set of momenta, we require $[J_i, J_j] = 0$; functions satisfying this condition are said to be 'in involution'.

of fixed $\mathbf{J}$ and varying $\mathbf{w}$ constitutes a 3-torus. The best set of labels J_i for this torus are the Poincaré invariants

$$\begin{aligned} J_i' &\equiv \frac{1}{2\pi}\oint_{\gamma_i} \mathbf{p}\cdot \mathrm{d}\mathbf{q} \\ &= \frac{1}{2\pi}\iint \sum_i \mathrm{d}p_i \mathrm{d}q_i, \end{aligned} \tag{3.5}$$

where the path γ_i is that followed when w_i is varied from 0 to 2π while holding everything else constant. (J_i' can be thought of as $1/2\pi$ times the area of torus's i^{th} cross-section—in 1-d the (J,w) system constitutes polar coordinates with $\frac{1}{2}r^2$ used as the radial variable.) Actually, $J_i' = J_i$. To see this, let $S(\mathbf{J},\mathbf{x})$ be the generating function of the transformation from Cartesian to action-angle coordinates. Then

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \quad ; \quad \mathbf{w} = \frac{\partial S}{\partial \mathbf{J}}, \tag{3.6}$$

so

$$\begin{aligned} J_i' &\equiv \frac{1}{2\pi}\oint_{\gamma_i} \frac{\partial S}{\partial \mathbf{q}}\cdot \mathrm{d}\mathbf{q} \\ &= \frac{\Delta S}{2\pi} \end{aligned} \tag{3.7}$$

But we also have (See Appendix II)

$$\Delta S = \oint_{\gamma_i} \mathbf{J}\cdot \mathrm{d}\mathbf{w} = 2\pi J_i.$$

We can use $S(\mathbf{J},\mathbf{q})$ to eliminate $\mathbf{p} = \partial S/\partial \mathbf{q}$ from H, expressing H as a function of $(\mathbf{J},\mathbf{q})$:

$$H(\mathbf{J},\mathbf{q}) = H\Big(\frac{\partial S}{\partial \mathbf{q}},\mathbf{q}\Big). \tag{3.8}$$

By moving on an orbit we can vary the q_i pretty much at will while holding constant the J_i. As we vary the q_i in this way H must remain constant at the energy E of the orbit in question. This suggests that we investigate the partial differential equation

$$H\Big(\frac{\partial S}{\partial \mathbf{q}},\mathbf{q}\Big) = E, \qquad \textbf{(Hamilton-Jacobi equation)}. \tag{3.9}$$

If we can solve this equation, we identify the arbitrary constants on which the solution $S(\mathbf{q})$ depends with functions of the J_i.

Exercise (2):

Show that for a harmonic oscillator of frequency ω the Hamilton-Jacobi equation reads

$$\Big(\frac{\mathrm{d}S}{\mathrm{d}x}\Big)^2 + m^2\omega^2 x^2 = 2mE.$$

Identify a new momentum P which allows S to be written

$$S(P,x)=(\theta+\tfrac{1}{2}\sin 2\theta)P \quad \text{where} \quad \theta(P,x)\equiv\arcsin\left(\sqrt{\frac{m\omega}{2P}}x\right).$$

Hence show that the action-angle coordinates of this system may be taken to be

$$\begin{aligned}P&\equiv\frac{1}{2m\omega}(p^2+m^2\omega^2x^2),\\ Q&\equiv\arctan(m\omega x/p).\end{aligned}$$

3.1 Delaunay variables

The H-J eqn for Kepler motion is

$$\begin{aligned}E&=\tfrac{1}{2}|\nabla S|^2-\frac{GM}{r}\\ &=\tfrac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial S}{\partial\theta}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial S}{\partial\phi}\right)^2\right]-\frac{GM}{r}\end{aligned}\tag{3.10}$$

We write $S(\mathbf{x})=S_r(r)+S_\theta(\theta)+S_\phi(\phi)$ and solve (3.10) by separation of variables. We find

$$\text{constant}\equiv L_z^2=\left(\frac{\partial S_\phi}{\partial\phi}\right)^2=p_\phi^2\tag{3.11a}$$

$$L^2-\frac{L_z^2}{\sin^2\theta}=\left(\frac{\partial S_\theta}{\partial\theta}\right)^2=p_\theta^2\tag{3.11b}$$

$$2E+2\frac{GM}{r}-\frac{L^2}{r^2}=\left(\frac{\partial S_r}{\partial r}\right)^2=p_r^2.\tag{3.11c}$$

Each of these 3 eqns is a relation of the form $p_i(q_i)$. The orbital torus is the 3-surface generated by varying the phase-space coordinates through all values compatible with these eqns. Since ϕ doesn't occur in (3.11a), it can take any value while p_ϕ is restricted to the single value L_z. Eqns (3.11b) and (3.11c) restrict θ and r to ranges and allow non-zero variation of p_θ and p_r. The orbit's inclination is just $i=\frac{\pi}{2}-\theta_{\rm min}$ and from eq (3.11b) we have that $\theta_{\rm min}=L_z/L$, so

$$i=\arccos(L_z/L).$$

The actions are

$$
\begin{aligned}
J_\phi &= \frac{1}{2\pi}\oint p_\phi(\phi)\,\mathrm{d}\phi = L_z \\
J_\theta &= \frac{1}{2\pi}\oint p_\theta(\theta)\,\mathrm{d}\theta \\
&= \frac{2}{\pi}\int_{\theta_{\min}}^{\pi/2}\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\,\mathrm{d}\theta = L - L_z \\
J_r &= \frac{1}{2\pi}\oint\sqrt{2E + \frac{2GM}{r} - \frac{L^2}{r^2}}\,\mathrm{d}r \\
&= \frac{GM}{\sqrt{2|E|}} - L.
\end{aligned}
\tag{3.12}
$$

The last equation can be written

$$
E = H = -\frac{G^2M^2}{2(J_r + J_\theta + J_\phi)^2}, \tag{3.13}
$$

from which it follows that $\omega_r = \omega_\theta = \omega_\phi = -2H/(J_r + J_\theta + J_\phi)$. It is this unusual double degeneracy that causes all Kepler orbits to close, and gives rise to specially complex behaviour when we perturb H_K.

Three of the 5 constants of motion of a Kepler orbit may be taken to be its three actions. The other two are constant by virtue of the system's degeneracy: this makes two differences in angles constant, for example $w_1 \equiv w_\phi - w_\theta$ and $w_2 \equiv w_\theta - w_r$. It is useful to make a canonical transformation to a set which includes these new angles. The generating function of this transformation is

$$
S' = (w_\phi - w_\theta)J_1 + (w_\theta - w_r)J_2 + w_rJ_3. \tag{3.14}
$$

Differentiating S' w.r.t. the old angles we discover the connection between the new and old actions:

$$
\begin{aligned}
J_\phi &= J_1 \\
J_\theta &= J_2 - J_1 \qquad \Rightarrow \qquad J_2 = J_\theta + J_\phi = L \\
J_r &= J_3 - J_2 \qquad\qquad\quad J_3 = J_r + J_\theta + J_\phi.
\end{aligned}
$$

Replacing the old actions by the new in our formulae, we have

$$
\begin{aligned}
H &= -\frac{G^2M^2}{2J_3^2} = -\frac{GM}{2a} \qquad \Rightarrow \qquad J_3 = \sqrt{GMa}, \\
e &= \sqrt{1 - \frac{L^2}{GMa}} = \sqrt{1 - J_2^2/J_3^2}, \\
i &= \arccos(J_1/J_2).
\end{aligned}
\tag{3.15}
$$

It remains only to relate w_1 and w_2 to Ω and ω (which they actually equal). We have $w_i = (\partial S/\partial J_i)$. Now replacing E, L and L_z in our expressions for $\partial S_\phi/\partial\phi$ etc we find

$$
S = \int J_1\mathrm{d}\phi - \int\sqrt{J_2^2 - \frac{J_1^2}{\sin^2\theta}}\,\mathrm{d}\theta + \int\sqrt{-\frac{G^2M^2}{J_3^2} + 2\frac{GM}{r} - \frac{J_2^2}{r^2}}\,\mathrm{d}r. \tag{3.16}
$$

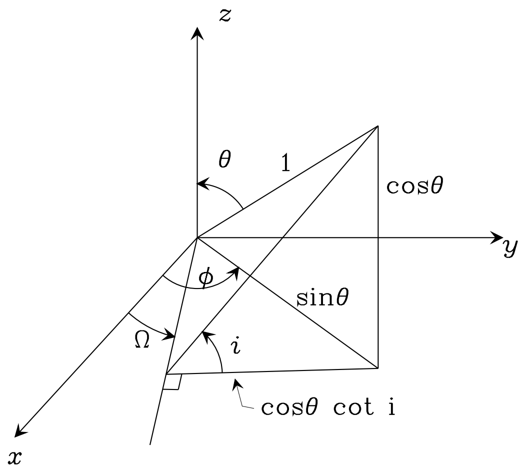
(We take the negative square root in S_θ because as planet passes through the *ascending* node, θ is decreasing and so $p_\theta < 0$.) Differentiating we get

$$w_1 = \frac{\partial S}{\partial J_1} = \phi + \int \frac{\mathrm{d}\theta}{\sin\theta\sqrt{\sin^2\theta\sec^2 i - 1}} \\ = \phi - u, \tag{3.17a}$$

where

$$\sin u \equiv \cot i \cot\theta. \tag{3.17b}$$

A figure demonstrates that the new variable u is actually $\phi - \Omega$ and thus that $w_1 = \Omega$:



Exercise (3):
By differentiating (3.16) show that $w_3 = G^2M^2t/J_3^3$. Obtain this result by another route.

Exercise (4):
Show that the integrals obtained by differentiating (3.16) w.r.t. J_2 evaluate to $\arcsin(\cos\theta/\sin i)$ and φ, where φ is the angle in the orbital plane between the direction of pericentre and the position of the planet. Hence conclude that $w_2 = \omega$, the argument of perihelion.

The variables (w_i, J_i) are called Delaunay variables. Let the true Hamiltonian of a planet be written $H = H_K + \epsilon H_1$, where H_K is the Kepler Hamiltonian defining the Delaunay variables and $\epsilon \ll 1$, then the equations of motion of the Delaunay variables are

$$\dot{J}_1 = -\epsilon\frac{\partial H_1}{\partial w_1} \quad ; \quad \dot{J}_2 = -\epsilon\frac{\partial H_1}{\partial w_2} \quad ; \quad \dot{J}_3 = -\epsilon\frac{\partial H_1}{\partial w_3} \\ \dot{w}_1 = \epsilon\frac{\partial H_1}{\partial J_1} \quad ; \quad \dot{w}_2 = \epsilon\frac{\partial H_1}{\partial J_2} \quad ; \quad \dot{w}_3 = -\frac{2H_K}{J_3} + \epsilon\frac{\partial H_1}{\partial J_3} \tag{3.18}$$

Thus the rates of change of five of these variables are of order ϵ, and to zeroth-order, the sixth increases linearly in time.

4 The disturbing function

The only snag with this elegant scheme is the tedium of expressing H_1 as a function of Delaunay variables. Since the Newtonian gravitational force is not velocity-dependent, H_1 is a function of the planet's spatial coordinates. Thus it is a function on three-dimensional real space. Unfortunately, Delaunay variables are coordinates for six-dimensional phase space, so we have to treat H_1 as a function in this bigger space. Since $\mathbf{x}$ is a periodic function of $\mathbf{w}$, H_1 can be expanded in a Fourier series:

$$H_1\big(\mathbf{x}(\mathbf{J},\mathbf{w};t)\big) = \sum_{\mathbf{n}} h_{\mathbf{n}}(\mathbf{J};t)e^{\mathrm{i}\mathbf{n}\cdot\mathbf{w}}, \tag{4.1}$$

where $\mathbf{n}$ runs over vectors with integer components.

Let's see how this works out in a simple case. Consider the disturbance of an asteroid m_a by Jupiter. The eqns of motion of the three bodies in the problem are

$$m_s\ddot{\mathbf{x}}_s = \frac{Gm_sm_J}{|\mathbf{x}_J-\mathbf{x}_s|^3}(\mathbf{x}_J-\mathbf{x}_s) + \frac{Gm_sm_a}{|\mathbf{x}_a-\mathbf{x}_s|^3}(\mathbf{x}_a-\mathbf{x}_s) \tag{4.2a}$$

$$m_J\ddot{\mathbf{x}}_J = -\frac{Gm_sm_J}{|\mathbf{x}_J-\mathbf{x}_s|^3}(\mathbf{x}_J-\mathbf{x}_s) + \frac{Gm_am_J}{|\mathbf{x}_a-\mathbf{x}_J|^3}(\mathbf{x}_a-\mathbf{x}_J) \tag{4.2b}$$

$$m_a\ddot{\mathbf{x}}_a = -\frac{Gm_sm_a}{|\mathbf{x}_a-\mathbf{x}_s|^3}(\mathbf{x}_a-\mathbf{x}_s) - \frac{Gm_am_J}{|\mathbf{x}_a-\mathbf{x}_J|^3}(\mathbf{x}_a-\mathbf{x}_J). \tag{4.2c}$$

We take suitable multiples of (4.2a) from (4.2b) and (4.2c) to obtain eqns of motion of the heliocentric vectors $\mathbf{r}_a \equiv \mathbf{x}_a - \mathbf{x}_s$ etc:

$$m_J\ddot{\mathbf{r}}_J = -\frac{G(m_s+m_J)m_J}{r_J^3}\mathbf{r}_J + \frac{Gm_am_J}{|\mathbf{r}_a-\mathbf{r}_J|^3}(\mathbf{r}_a-\mathbf{r}_J) - \frac{Gm_Jm_a}{r_a^3}\mathbf{r}_a, \quad \text{(4.3a))}$$

$$m_a\ddot{\mathbf{r}}_a = -\frac{G(m_s+m_a)m_a}{r_a^3}\mathbf{r}_a - \frac{Gm_am_J}{|\mathbf{r}_a-\mathbf{r}_J|^3}(\mathbf{r}_a-\mathbf{r}_J) - \frac{Gm_am_J}{r_J^3}\mathbf{r}_J. \tag{4.3b}$$

Eqn (4.3b) can be written

$$\ddot{\mathbf{r}}_a = -\frac{\partial}{\partial\mathbf{r}_a}\left[G\Big(-\frac{m_s+m_a}{r_a} - \frac{m_J}{|\mathbf{r}_a-\mathbf{r}_J|} + m_J\frac{\mathbf{r}_a\cdot\mathbf{r}_J}{r_J^3}\Big)\right]. \tag{4.4}$$

Exercise (5):

Generalize (4.4) to the case of n planets.

The left side of (4.4) plus the first term on the right are the eqn of motion for Kepler motion around a Sun fixed at the origin. The other two terms in the potential on the right can be considered to constitute a perturbing potential. So in the Hamiltonian formulation of this problem the zeroth-order Hamiltonian is $H_K = \frac{1}{2}\dot{r}_a^2 - G(m_s+m_a)/r_a$ and the perturbing Hamiltonian is

$$H_1 = -Gm_J\Big(\frac{1}{|\mathbf{r}_a-\mathbf{r}_J|} - \frac{\mathbf{r}_a\cdot\mathbf{r}_J}{r_J^3}\Big). \tag{4.5}$$

Minus H_1 is called the **disturbing function**.

5 Perturbation Theory

Consider first the simple case in which both Jupiter and the asteroid are on circular orbits in the ecliptic. Then

$$H_1 = -\frac{Gm_J}{a_J}\Big((1+\alpha^2-2\alpha\cos\varphi)^{-1/2}-\alpha\cos\varphi\Big), \tag{5.1}$$

where $\alpha \equiv a_a/a_J$ and φ is the angle between $\mathbf{r}_a$ and $\mathbf{r}_J$. The radical in (5.1) is often expanded in a Fourier series to give

$$H_1 = -\frac{Gm_J}{a_J}\Big(\sum_{m=-\infty}^{\infty}\tfrac{1}{2}b_{1/2}^{(m)}\cos m\varphi-\alpha\cos\varphi\Big). \tag{5.2}$$

The functions $b_{1/2}^{(m)}(\alpha)$ are known as **Laplace coefficients** and are tabulated, e.g., in Brouwer & Clements.

$w_1 = \Omega$ is redundant for orbits confined to the ecliptic and may be set to zero. Then $\varphi = w_2 + w_3 - w_2' - w_3'$, where the primes denote Jupiter's coordinates. The natural thing to do now is to solve for the evolution of the unprimed coordinates with the primed coordinates replaced by their unperturbed values, $w_2' = $ constant, $w_3' = \nu_J t$, where $\nu_J \equiv 2\pi/T_J$ is Jupiter's angular frequency. For example,

$$\dot J_2 = \frac{Gm_J}{a_J}\frac{\partial}{\partial w_2}\Big(\sum_m b_{1/2}^{(m)}\cos\big(m(w_2+w_3-\nu_J t)\big)-\alpha\cos(w_2+w_3-\nu_J t)\Big) \tag{5.3}$$

There is clearly no difficulty integrating these coupled equations. The solutions will show the J_i to wiggle around without going anywhere much.

The problem becomes considerably more interesting and complex when we allow for non-zero eccentricity. Heliocentric distance r is related to w_3 by

$$r = a(1-e\cos\eta) \quad \text{where the \textbf{eccentric anomaly} } \eta \text{ solves} \quad \eta - e\sin\eta = w_3. \tag{5.4}$$

Using this in (4.5)

$$\begin{aligned} H_1 = -\frac{Gm_J}{a_J}\Big\{&\big[(1-e'\cos\eta')^2+\alpha^2(1-e\cos\eta)^2 \\ &-2\alpha(1-e'\cos\eta')(1-e\cos\eta)\cos\varphi\big]^{-1/2}-\frac{\alpha(1-e\cos\eta)}{(1-e'\cos\eta')^2}\cos\varphi\Big\}. \end{aligned} \tag{5.5}$$

Now imagine replacing φ with $w_2 + w_3 - \nu_J t$, substituting from (5.4) for η and η' in terms of w_3 and $\nu_J t$, and expanding the right side in a Fourier series in w_2, w_3 and t. The products of circular functions will clearly generate all sorts of sum and difference frequencies, so H_1 will be of the form

$$H_1 = -\frac{Gm_J}{a_J}\sum_{\mathbf{n}} h_{\mathbf{n}}(e,e')e^{\mathrm{i}(n_1\nu_J t+n_2 w_2+n_3 w_3)}. \tag{5.6}$$

The eqns of motion of J_2 and w_2 are now

$$\begin{aligned}\dot{J}_2 &= \frac{Gm_J}{a_J}\sum_{\mathbf{n}} \mathrm{i}n_2 h_{\mathbf{n}}(e,e')e^{\mathrm{i}(n_1\nu_J t+n_2w_2+n_3w_3)}\\ \dot{w}_2 &= -\frac{Gm_J}{a_J}\sum_{\mathbf{n}}\frac{\partial h_{\mathbf{n}}}{\partial e}\frac{\partial e}{\partial J_2}e^{\mathrm{i}(n_1\nu_J t+n_2w_2+n_3w_3)}.\end{aligned} \tag{5.7}$$

The n_i are expected to range over all positive and negative integers, and in the unperturbed motion w_2 is constant and $w_3 = \nu_a t + \phi_0$, where ϕ_0 is the asteroid's phase with respect to Jupiter. So for an asteroid with $\nu_a = 3\nu_J$ the right side of (5.7) will contain 'long-period terms' such as those with $\mathbf{n}$ of the form $\mathbf{n} = (3, n_2, -1)$—that is, it will contain terms in which the exponent is constant in the unperturbed case. These terms cause disaster when we integrate up (5.7) with the right side evaluated along the unperturbed orbit:

$$J_2 = \frac{Gm_J}{a_J}\sum_{\mathbf{n}}\frac{n_2 h_{\mathbf{n}}(e,e')}{n_1\nu_J+n_3\nu_a}e^{\mathrm{i}t(n_1\nu_J+n_3\nu_a)}e^{\mathrm{i}(n_2w_2+n_3\phi_0)} + \text{constant}. \tag{5.8}$$

The local spot of bother caused by the near vanishing of some of the denominators $n_1\nu_J + n_3\nu_a$ is called the 'small divisor problem'. Despite the efforts of the greatest mathematicians up to and including H. Poincaré, it was only mastered, and then but partially, in the 1950-60s by Kolmogorov, Arnold & Moser (the 'KAM' theorem).

5.1 Pendulum equations

The physical origin of the small denominator problem is that the forces described by long-period terms act in one sense for long enough to cause the orbit to deviate significantly from its unperturbed form. So these terms which cause grief are actually the interesting terms. The basic idea of celestial mechanics is to neglect the short period terms on the grounds that they average out to zero, and to concentrate on the long-period terms.

Suppose we are interested in an asteroid along whose orbit $N_1\nu_J + N_3\nu_a \simeq 0$. To keep things simple let's set $J_2 = w_2 = 0$ and concentrate on the evolution of (J_3, w_3). We define a new variable

$$\psi \equiv N_1\nu_J t + N_3 w_3 \tag{5.9}$$

and discard all terms in the sum (5.6) for H_1 except the zero-frequency term and the terms (at positive and negative frequency) that comprise $\cos(\psi)$:[2]

$$H_1 \simeq h_0 + h_{\rm res}\cos\psi. \tag{5.10}$$

This discarding of the short-period terms is called 'averaging the Hamiltonian'. If we are to use ψ as a coordinate, we need to know what its conjugate momentum J_ψ is. The generating function for the transformation $(w_3, J_3) \leftrightarrow (\psi, J_\psi)$ is

$$S(w_3, J_\psi) = (N_1\nu_J t + N_3 w_3)J_\psi \quad\Rightarrow\quad \begin{aligned}\psi &= \frac{\partial S}{\partial J_\psi} = N_1\nu_J t + N_3 w_3,\\ J_3 &= \frac{\partial S}{\partial w_3} = N_3 J_\psi.\end{aligned} \tag{5.11}$$

[2] We can arrange for the phase of the cosine to be zero by choosing our origin of time intelligently.

Since the generating function contains explicit t-dependence in the new variables, the Hamiltonian (5.10) is

$$\begin{aligned} H_1(J_\psi,\psi) &= H_1(J_3,w_3,t) + \frac{\partial S}{\partial t} \\ &= h_0 + N_1\nu_J J_\psi + h_{\rm res}\cos\psi. \end{aligned} \tag{5.12}$$

Neither the unperturbed Hamiltonian $H_0(J_3) = H_0(N_3 J_\psi)$ nor the new H_1 contains explicit t-dependence. So the motion is confined to surfaces of constant

$$\begin{aligned} H(J_\psi,\psi) &= H_0 + h_0 + N_1\nu_J J_\psi + h_{\rm res}\cos\psi \\ &= \alpha(J_\psi) - \beta(J_\psi)\cos\psi. \end{aligned} \tag{5.13}$$

We assume that $\beta > 0$—if it isn't we define $\psi' = \psi + \pi$ and then $\beta' = -\beta$ will be positive. Note that $\beta/|\alpha| \simeq m_J a_a/m_s a_J$ is small.

For some value J_0 of J_ψ near the asteroid's action the unperturbed orbit is perfectly resonant, i.e. $0 = \dot\psi = \partial H_0/\partial J_\psi = {\rm d}\alpha/{\rm d}J_\psi$. So Taylor expanding α in powers of $\Delta \equiv J_\psi - J_0$ we have

$$H(\Delta,\psi) = A + \tfrac{1}{2}B\Delta^2 - \beta\cos\psi, \tag{5.14}$$

where A and B are constants. Since β is small, $\beta(J_\psi) \simeq \beta(J_0)$ to a sufficient approximation and the eqns of motion are approximately

$$\left.\begin{aligned} \dot\psi &= B\Delta \\ \dot\Delta &= -\beta\sin\psi \end{aligned}\right. \quad\Rightarrow\quad \ddot\psi = -B\beta\sin\psi. \tag{5.15}$$

This is the eqn of motion of a pendulum. Two qualitatively different motions are possible: either the pendulum circulates in a constant sense because it has enough energy to carry it over top dead centre, or it lacks this critical energy and swings to and fro. Quantitatively it circulates when

$$\text{K.E. at bottom} = \tfrac{1}{2}\dot\psi^2 > 2B\beta = \text{P.E. difference between top and bottom.} \tag{5.16}$$

When ψ is oscillating around $\psi = 0$, one says that the asteroid is **librating** in w_3. There are asteroids trapped in this way for $N_1/N_2 = 1$ (**Trojan asteroids**) and 3/2 (**Hilda asteroids**). Other important resonances are marked by **Kirkwood gaps**.

Exercise (6):

Show that the Hamiltonian of a pendulum of length l making angle ψ with the vertical is $H_p = \frac{p^2}{2l^2} - gl\cos\psi$. Plot curves of constant H is phase space.

Now let's unfreeze (J_2, w_2). We again discard from the eqns of motion (5.7) all but the slowest-varying terms. Since w_2 is constant along an unperturbed orbit, in these terms $n_1 = N_1$ and $n_3 = N_3$ as above. Of the remaining terms, that with $n_2 = 0$ is the most important because this causes w_2 to drift steadily. In fact, w_2 increases as Jupiter depresses the frequency of the asteroid's radial oscillations below that of its rotation about the Sun, causing perihelion to precess forward. That is, to first-order $\omega_2 > 0$.

Only terms with $n_2 \neq 0$ contribute to the eqn for $\dot{J}_2$. These have constant exponents when

$$n_1\nu_J + n_2\omega_2 + n_3\nu_a = 0, \tag{5.17}$$

a condition satisfied for slightly different actions than those satisfying $n_1\nu_J + n_3\nu_a = 0$. The method used above to handle the case of fixed (J_2, w_2) can be used here too: we now define

$$\psi \equiv N_1\nu_J t + N_2 w_2 + N_3 w_a \tag{5.18}$$

and transform to angles which include ψ. On throwing away rapidly varying terms we find that ψ again satisfies a pendulum eqn.

Notice that the resonance condition (5.17) can be satisfied for any value of n_2 because ω_2 is small. However, the $h_{\mathbf{n}}$ are very small for large $|n_2|$ unless e is large.

The analysis of the motion of (J_1, w_1) proceeds similarly to that reviewed for (J_2, w_2). However, the first-order contribution to ω_1 is negative; Jupiter causes the line of nodes to precess backwards since it raises the vertical frequency above the circular frequency. Hence vertical resonance occurs on rather different orbits from which satisfy (5.17).

Each stable resonant orbit is surrounded by a family of trapped librating orbits—the stable orbit is that on which $\psi = 0$ and the librating orbits are those on which the pendulum swings to and fro. Chaos ensues when two or more families of neighbouring resonances claim the alegance of the same orbits, that is, when resonances 'overlap'. Whether or not resonance overlap occurs depends on (i) the magnitude of the $h_{\mathbf{n}}$, (which determine the width of the resonances), and (ii) the nearness of the renonaces.

The ring systems of Saturn and Uranus are profoundly affected by the flattenings of the planets, which depress ω_1 below zero and raise ω_2 above zero.

Exercise (7):

Model the effect of the planet's flattening by adding to the regular monopole potential $-GM/r$ a quadrupole term $\alpha GM(3z^2 - r^2)/r^5$. Verify these statements about ω_1 and ω_2 by showing that the radial and vertical epicycle frequencies are given by

$$\kappa_r^2 = \left(1 - 3\frac{\alpha^2}{r^2}\right)\frac{GM}{r^3} \quad ; \quad \kappa_z^2 = \left(1 + 9\frac{\alpha^2}{r^2}\right)\frac{GM}{r^3}.$$

Around Saturn the resonances are sufficiently well separated for order to be the rule. Around Uranus many important resonances overlap and chaos is widespread.