Quantum Mechanics: Set 2

1. Show that the creation and annihilation operators of a harmonic oscillator satisfy $[A, (A^\dagger)^n] = n(A^\dagger)^{n-1}$. Let α be a complex number and $|0\rangle$ be the ground state. Show that the (un-normalised) state $|\alpha\rangle$ defined by

$$|\alpha\rangle = \exp(\alpha A^\dagger)|0\rangle \quad \text{satisfies} \quad A|\alpha\rangle = \alpha|\alpha\rangle.$$

By solving this equation in the position representation, show that

$$\langle x|\alpha\rangle = N \exp\left[-\frac{m\omega x^2}{2\hbar} + \sqrt{\frac{2m\omega}{\hbar}}\,\alpha x\right],$$

where N is a normalisation constant that need not be determined. Sketch as a function of x the probability density $|\langle x|\alpha\rangle|^2$.

2. Show that the spin-one matrix S_x is related to the matrix $\mathbf{J}_x$ that effects classical rotations of vectors around the x axis by $S_x = \mathbf{B}\mathbf{J}_x\mathbf{B}^\dagger$, where

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix}$$

Hence show that S_x would be the matrix associated with rotations of vectors about the x axis if we used new coordinates $X_1 = (x + \mathrm{i}y)/\sqrt{2}$, $X_2 = z$, $X_3 = -(x - \mathrm{i}y)/\sqrt{2}$.

3. Write down the spin matrix S_z for $s = 3/2$ and show that in this case the other spin matrices are

$$S_x = \hbar \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \quad ; \quad S_y = \hbar \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}\mathrm{i} & 0 & 0 \\ \frac{\sqrt{3}}{2}\mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & -\frac{\sqrt{3}}{2}\mathrm{i} \\ 0 & 0 & \frac{\sqrt{3}}{2}\mathrm{i} & 0 \end{pmatrix}$$

4. Consider states of well defined S_z for two gyros with total spin quantum numbers j_1 and j_2. Then there are $(2j_1 + 1)(2j_2 + 1)$ states of the form $|j_1, m_1\rangle|j_2, m_2\rangle$. Show that there are exactly this number of accessible states $|j, m\rangle$ of well defined angular momentum for a box that contains the gyros.

5. The fundamental principle of statistical physics is that every quantum state of a system has equal a priori probability. Hence, the a priori probability that the above box has total angular momentum $j\hbar$ is

$$P(j) = \frac{2j+1}{(2j_1+1)(2j_2+1)}.$$

Consider now the magnitude J of the total angular momentum of a box that contains two classical gyros, of spin J_1 and J_2. If the orientation of the second gyro relative to the first is completely unknown, show that the probability $\mathrm{d}P$ that J lies in $(J+\mathrm{d}J, J)$ is

$$\mathrm{d}P = \frac{J\,\mathrm{d}J}{2J_1J_2}.$$

Show that in the limit of large j_1, j_2 the quantum result agrees with this classical expression.

6. Two spin-half particles interact through the Hamiltonian

$$H = K\mathbf{S}_1.\mathbf{S}_2,$$

where K is a constant and S_{1i} is the ith spin operator of the first particle, etc. Show that the total spin operator $S^2 = |\mathbf{S}_1 + \mathbf{S}_2|^2$ commutes with H. State two implications of this result.

By determining the matrix elements of H in the representation in which S^2 and S_z is diagonal, show that the eigenvalues of H are $\frac{1}{4}K\hbar^2$ and $-\frac{3}{4}K\hbar^2$. Which eigenvalue is degenerate? [Hint: use $\mathbf{S}_1.\mathbf{S}_2 = \frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z})$.]

7. Two spin-$\frac{1}{2}$ particles, with spin operators $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$, have a Hamiltonian

$$H = \alpha\mathbf{S}^{(1)}.\mathbf{S}^{(2)} + \beta\mathbf{B}.(\mathbf{S}^{(1)} - \mathbf{S}^{(2)})$$

where $\mathbf{B}$ represents an applied external static magnetic flux density. Show that H may be rewritten as a function of $S_\pm^{(1)}$, $S_z^{(1)}$, $S_\pm^{(2)}$, and $S_z^{(2)}$. Using the basis states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ (where $|\uparrow\uparrow\rangle$ corresponds to the state with both spins up etc.) find a matrix representation for H. Hence find the energy eigenvalues and comment on their behaviour as $\mathbf{B}$ tends to zero.

8. (TO 1996) In a certain representation, the matrix representing the Hamiltonian for a three-state system has the form

$$H = H_0 + V$$

where

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \quad \text{and} \quad V = V_0 \begin{pmatrix} 0 & \epsilon & \epsilon^2 \\ \epsilon & 0 & 0 \\ \epsilon^2 & 0 & 0 \end{pmatrix},$$

and where $|\epsilon| \ll 1$.

(a) Consider first the case $E_1 = E_2$, and $|\epsilon V_0/(E_3 - E_2)| \ll 1$. Use (without proof) the results of first order degenerate perturbation theory to obtain the energy eigenvalues correct to order ϵ.

(b) Now consider the different case $E_2 = E_3$, and $|\epsilon V_0/(E_1 - E_2)| \ll 1$. Obtain the energy eigenvalues correct to order ϵ^2.

Why is it not necessary to use degenerate perturbation theory in this case, even though $E_2 = E_3$?

(c) Obtain the exact energy eigenvalues in case (b), and by expanding them up to second order in ϵ, recover the perturbation theory result.

9. A system has two eigenstates i, j separated in energy by $\hbar\omega_{ij}$. It is subject to a small time-independent perturbation for a time T. The perturbation has matrix elements $V_{ji} = V_{ij}^*$ between these eigenstates. Show that if the system is initially in state i, the probability of a transition to state j is approximately

$$P_{ij} = 4|V_{ij}|^2 \frac{\sin^2(\omega_{ij}T/2)}{(\hbar\omega_{ij})^2}.$$

A neutral particle with spin $\frac{1}{2}$ and magnetic moment μ is travelling at speed v in a region of uniform magnetic field with flux density B. Over a small length ℓ of its path an additional flux density b is applied at right angles to B. As a result of the motion of the particle, its spin wavefunction satisfies the Schrödinger equation with a time-dependent Hamiltonian $H(t)$ given by

$$H(t) = \begin{cases} -\mu(B\sigma_z + b\sigma_x) & 0 < t < \ell/v \\ -\mu B\sigma_z & \text{otherwise.} \end{cases} \qquad (\dagger)$$

The system originally has spin $+\frac{1}{2}$ with respect to the direction of B. Find the probability that it makes a transition to the state with opposite spin:

(i) by assuming $b \ll B$ and using ($\dagger$);

(ii) by finding the exact evolution of the state.