Introduction to Quantum Mechanics MT 2009
Problems 2 (Weeks 7–8)

2.1 Write down the time-independent (TISE) and the time-dependent (TDSE) Schrödinger equations. Is it necessary for the wavefunction of a system to satisfy the TDSE? Under what circumstances does the wavefunction of a system satisfy the TISE?

2.2 Why is the TDSE first-order in time, rather than second-order like Newton's equations of motion?

2.3 A particle is confined in a potential well such that its allowed energies are $E_n = n^2\mathcal{E}$, where $n = 1, 2, \ldots$ is an integer and $\mathcal{E}$ a positive constant. The corresponding energy eigenstates are $|1\rangle$, $|2\rangle$, $\ldots$, $|n\rangle, \ldots$. At $t = 0$ the particle is in the state

$$|\psi(0)\rangle = 0.2|1\rangle + 0.3|2\rangle + 0.4|3\rangle + 0.843|4\rangle. \tag{2.1}$$

a. What is the probability, if the energy is measured at $t = 0$ of finding a number smaller than $6\mathcal{E}$?
b. What is the mean value and what is the rms deviation of the energy of the particle in the state $|\psi(0)\rangle$?
c. Calculate the state vector $|\psi\rangle$ at time t. Do the results found in (a) and (b) for time t remain valid for arbitrary time t?
d. When the energy is measured it turns out to be $16\mathcal{E}$. After the measurement, what is the state of the system? What result is obtained if the energy is measured again?

2.4 Let $\psi(x)$ be a properly normalised wavefunction and Q an operator on wavefunctions. Let $\{q_r\}$ be the spectrum of Q and $\{u_r(x)\}$ be the corresponding correctly normalised eigenfunctions. Write down an expression for the probability that a measurement of Q will yield the value q_r. Show that $\sum_r P(q_r|\psi) = 1$. Show further that the expectation of Q is $\langle Q\rangle \equiv \int_{-\infty}^{\infty} \psi^* \hat{Q}\psi \, dx$.[1]

2.5 Find the energy of neutron, electron and electromagnetic waves of wavelength 0.1 nm.

2.6 Neutrons are emitted from an atomic pile with a Maxwellian distribution of velocities for temperature 400 K. Find the most probable de Broglie wavelength in the beam.

2.7 A beam of neutrons with energy E runs horizontally into a crystal. The crystal transmits half the neutrons and deflects the other half vertically upwards. After climbing to height H these neutrons are deflected through 90° onto a horizontal path parallel to the originally transmitted beam. The two horizontal beams now move a distance L down the laboratory, one distance H above the other. After going distance L, the lower beam is deflected vertically upwards and is finally deflected into the path of the upper beam such that the two beams are co-spatial as they enter the detector. Given that particles in both the lower and upper beams are in states of well-defined momentum, show that the wavenumbers k, k' of the lower and upper beams are related by

$$k' \simeq k\left(1 - \frac{m_{\rm n} g H}{2E}\right). \tag{2.2}$$

In an actual experiment (R. Colella et al., 1975, Phys. Rev. Let., 34, 1472) $E = 0.042\,\text{eV}$ and $LH \sim 10^{-3}\,\text{m}^2$ (the actual geometry was slightly different). Determine the phase difference between the two beams at the detector. Sketch the intensity in the detector as a function of H.

2.8 A three-state system has a complete orthonormal set of states $|1\rangle, |2\rangle, |3\rangle$. With respect to this basis the operators H and B have matrices

$$H = \hbar\omega\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad B = b\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \tag{2.3}$$

where ω and b are real constants.
a. Are H and B Hermitian?
b. Write down the eigenvalues of H and find the eigenvalues of B. Solve for the eigenvectors of both H and B. Explain why neither matrix uniquely specifies its eigenvectors.
c. Show that H and B commute. Give a basis of eigenvectors common to H and B.

[1] In the most elegant formulation of qantum mechanics, this last result is the basic postulate of the theory, and one *derives* other rules for the physical interpretation of the q_n, a_n etc. from it – see J. von Neumann, *Mathematical Foundations of Quantum Mechanics.*

2.9 Given that A and B are Hermitian operators, show that $\mathrm{i}[A, B]$ is a Hermitian operator.

2.10 Given a ordinary function $f(x)$ and an operator R, the operator $f(R)$ is defined to be

$$f(R) = \sum_i f(r_i)|r_i\rangle\langle r_i|, \tag{2.4}$$

where r_i are the eigenvalues of R and $|r_i\rangle$ are the associated eigenkets. Show that when $f(x) = x^2$ this definition implies that $f(R) = RR$, that is, that operating with $f(R)$ is equivalent to applying the operator R twice. What bearing does this result have in the meaning of e^R?

2.11 Show that if there is a complete set of mutual eigenkets of the Hermitian operators A and B, then $[A, B] = 0$. Explain the physical significance of this result.

2.12 Given that for any two operators $(AB)^\dagger = B^\dagger A^\dagger$, show that

$$(ABCD)^\dagger = D^\dagger C^\dagger B^\dagger A^\dagger. \tag{2.5}$$