Introduction to Quantum Mechanics MT 2009

Problems 1 (weeks 5-6 of MT)

1.1 What physical phenomenon requires us to work with probability amplitudes rather than just with probabilities, as in other fields of endeavour?

1.2 What properties cause complete sets of amplitudes to constitute the elements of a vector space?

1.3 V' is the adjoint space of the vector space V. For a mathematician, what objects comprise V'?

1.4 In quantum mechanics, what objects are the members of the vector space V? Give an example for the case of quantum mechanics of a member of the adjoint space V' and explain how members of V' enable us to predict the outcomes of experiments.

1.5 Given that $|\psi\rangle = e^{i\pi/5}|a\rangle + e^{i\pi/4}|b\rangle$, express $\langle\psi|$ as a linear combination of $\langle a|$ and $\langle b|$.

1.6 What properties characterise the bra $\langle a|$ that is associated with the ket $|a\rangle$?

1.7 An electron can be in one of two potential wells that are so close that it can "tunnel" from one to the other. Its state vector can be written

$$|\psi\rangle = a|A\rangle + b|B\rangle, \tag{1.1}$$

where $|A\rangle$ is the state of being in the first well and $|B\rangle$ is the state of being in the second well and all kets are correctly normalised. What is the probability of finding the particle in the first well given that: (a) $a = i/2$; (b) $b = e^{i\pi}$; (c) $b = \frac{1}{3} + i/\sqrt{2}$?

1.8 An electron can "tunnel" between potential wells that form a chain, so its state vector can be written

$$|\psi\rangle = \sum_{-\infty}^{\infty} a_n|n\rangle, \tag{1.2a}$$

where $|n\rangle$ is the state of being in the n^{th} well, where n increases from left to right. Let

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{-i}{3}\right)^{|n|/2} e^{in\pi}. \tag{1.2b}$$

a. What is the probability of finding the electron in the n^{th} well?
b. What is the probability of finding the electron in well 0 or anywhere to the right of it?

1.9 How is a wave-function $\psi(x)$ written in Dirac's notation? What's the physical significance of the complex number $\psi(x)$ for given x?

1.10 Let Q be an operator. Under what circumstances is the complex number $\langle a|Q|b\rangle$ equal to the complex number $(\langle b|Q|a\rangle)^*$ for any states $|a\rangle$ and $|b\rangle$?

1.11 Let Q be the operator of an observable and let $|\psi\rangle$ be the state of our system.
a. What are the physical interpretations of $\langle\psi|Q|\psi\rangle$ and $|\langle q_n|\psi\rangle|^2$, where $|q_n\rangle$ is the n^{th} eigenket of the observable Q and q_n is the corresponding eigenvalue?
b. What is the operator $\sum_n |q_n\rangle\langle q_n|$, where the sum is over all eigenkets of Q? What is the operator $\sum_n q_n|q_n\rangle\langle q_n|$?
c. If $u_n(x)$ is the wavefunction of the state $|q_n\rangle$, write dow an integral that evaluates to $\langle q_n|\psi\rangle$.

1.12 What does it mean to say that two operators commute? What is the significance of two observables having mutually commuting operators?

Given that the commutator $[P, Q] \neq 0$ for some observables P and Q, does it follow that for all $|\psi\rangle \neq 0$ we have $[P, Q]|\psi\rangle \neq 0$?

1.13 Let $\psi(x, t)$ be the correctly normalised wavefunction of a particle of mass m and potential energy $V(x)$. Write down expressions for the expectation values of (a) x; (b) x^2; (c) the momentum p_x; (d) p_x^2; (e) the energy.

What is the probability that the particle will be found in the interval (x_1, x_2)?

1.14 A system has a time-independent Hamiltonian that has spectrum $\{E_n\}$. Prove that the probability P_k that a measurement of energy will yield the value E_k is is time-independent. Hint: you can do this either from Ehrenfest's theorem, or by differentiating $\langle E_k|\psi\rangle$ w.r.t. t and using the TDSE.

1.15 A particle moves in the potential $V(\mathbf{x})$ and is known to have energy E_n. (a) Can it have well defined momentum for some particular $V(\mathbf{x})$? (b) Can the particle simultaneously have well-defined energy and position?

1.16 The states $\{|1\rangle, |2\rangle\}$ form a complete orthonormal set of states for a two-state system. With respect to these basis states the operator σ_y has matrix

$$\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{1.3}$$

Could σ be an observable? What are its eigenvalues and eigenvectors in the $\{|1\rangle, |2\rangle\}$ basis? Determine the result of operating with σ_y on the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle). \tag{1.4}$$

1.17 Prove for any four operators A, B, C, D that

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC. \tag{1.5}$$

Explain the similarity with the rule for differentiating a product.

1.18 Show that a classical harmonic oscillator satisfies the virial equation $2\langle \mathrm{KE}\rangle = \alpha\langle \mathrm{PE}\rangle$ and determine the relevant value of α.

1.19 A classical fluid of density $\rho(\mathbf{x})$ flows with velocity $\mathbf{v}(\mathbf{x})$. By differentiating with respect to time the mass $m \equiv \int_V \mathrm{d}^3\mathbf{x}\,\rho$ contained in an arbitrary volume V, show that conservation of mass requires that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0. \tag{1.6}$$

Hint: the flux of matter at any point is $\rho\mathbf{v}$ and the integral of this flux over the boundary of V must equal the rate of accumulation of mass within V.

$\mathbf{J}$ is defined to be

$$\mathbf{J}(\mathbf{x}) \equiv \frac{\mathrm{i}\hbar}{2m}\Big(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi\Big), \tag{1.7}$$

where $\psi(\mathbf{x})$ is the wavefunction of a spinless particle of mass m. Working from the TDSE, show that

$$\frac{\partial|\psi|^2}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0. \tag{1.8}$$

Give a physical interpretation of this result.

Show that when we write the wavefunction in amplitude-modulus form, $\psi = |\psi|\mathrm{e}^{\mathrm{i}\theta}$,

$$\mathbf{J} = |\psi|^2\frac{\hbar\boldsymbol{\nabla}\theta}{m}. \tag{1.9}$$

Interpret this result physically. Given that $\psi = A\mathrm{e}^{\mathrm{i}(kz-\omega t)} + B\mathrm{e}^{-\mathrm{i}(kz+\omega t)}$, where A and B are constants, show that

$$\mathbf{J} = v\big(|A|^2 - |B|^2\big)\,\hat{\mathbf{z}}, \tag{1.10}$$

where $v = \hbar k/m$. Interpret the result physically.