Kinetic theory of self-gravitating systems

James Binney

1

Introduction

The aim of these 8 lectures is to show how the ideas introduced earlier in the second section of the course in connection with electrostatic plasmas can be extended to stellar systems, with sometimes surprising results. This is rather a small, even niche, corner of stellar dynamics, but an intriguing one that fits neatly with the remainder of this course. Appendix B attempts to give a broader, more physically oriented overview of the kinetics of stellar systems. That material should provide a clearer understanding of the physical principles that underlie the mathematical physics to which the course itself is confined. Further supplementary material can be found in *Galactic Dynamics*, Binney & Tremaine, PUP (2008) and in a review arXiv1309.2794 (NewAR, 57, 29). Any fan of recorded lectures can try the lectures at http://iactalks.iac.es/talks/view/329.

1.1 What differentiates stellar and electrostatic plasmas?

The key differences between a gravitational plasma and an electrostatic one are:

- Gravity, being an even-spin theory (e.g. A. Zee, *Quantum Field theory in a Nutshell*, Princeton University Press) endows all particles with the same sign of charge.
- Consequently, self-gravitating systems are globally inhomogeneous rather than inhomogeneous only on scales smaller than the Debye length.

Inhomogeneity is a major setback: when a system is translationally invariant, group theory guarantees that the linear equations governing small disturbances have a complete set of solutions of the form $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ and to understand the dynamics of disturbances we only have to determine the dispersion relation. When a system is inhomogeneous, we have to solve for the form of the eigenfunctions before we can look for the dispersion relation.

We do this in two stages. First we imagine a system in which the masses of particles are smeared through volumes that extend just a bit further than the local inter-particle distance. This system has a pretty smooth density distribution $\rho(\mathbf{x})$ and consequently a very smooth gravitational potential

$$\Phi(\mathbf{x}) = G\int \mathrm{d}^3\mathbf{x}'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{1.1}$$

Next, we study orbits in this smooth potential.

1.2 Angle-action variables

If you numerically integrate orbits in potentials similar to those of star clusters and galaxies and Fourier decompose the resulting time series $x(t)$, $y(t)$, etc, you generally find that these series are **quasiperiodic**.[1] That is, their Fourier decompositions are of the form

$$x(t) = \sum_{\mathbf{n}} X_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega} t}, \tag{1.2}$$

where the $2d$ or $3d$ vectors $\mathbf{n}$ have integer components and the vector $\boldsymbol{\Omega}$ is made up of 2 or 3 frequencies that are characteristic of the orbit.[2] From the quasiperiodic nature of $x(t)$ it can be shown (see V.I. Arnold *Mathematical Methods of Classical Mechanics* Springer) that the orbit admits at least as many independent **integrals of motion** $I(\mathbf{x}, \mathbf{v})$ as it has degrees of freedom. That is, there are at least 2 or 3 (depending on whether or not the orbit is confined to a plane) independent functions on phase space such that

$$\frac{\mathrm{d}}{\mathrm{d}t} I[\mathbf{x}(t), \mathbf{v}(t)] = 0. \tag{1.3}$$

In a time-independent potential, H is always an integral of motion, and in an axisymmetric potential the appropriate component of angular moment is always another integral. The non-trivial numerical result is that there is almost always a third integral of motion of unknown functional form.

Given a set of integrals of motion I_i, any function $J_i(I_1, I_2, I_3)$ of three variables provides another integral. Given this choice, it's natural to ask whether a set of integrals can be found that can be complemented by canonically-conjugate variables, θ_i. For if we had a system of canonical coordinates $(\boldsymbol{\theta}, \mathbf{J})$ such that the momenta were constant, half of Hamilton's equations would read

$$0 = \dot{J}_i = -\frac{\partial H}{\partial \theta_i}. \tag{1.4}$$

That is, these equations of motion establish that the Hamiltonian, and its derivatives, are functions of the J_i only and are therefore constant on each orbit. The other equations of motion are now trivially solved:

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} = \Omega_i(\mathbf{J}) \quad \text{a constant} \quad \Rightarrow \quad \theta_i(t) = \theta_i(0) + \Omega_i t. \tag{1.5}$$

So in the $(\boldsymbol{\theta}, \mathbf{J})$ coordinate system dynamics becomes trivial. The magic integrals J_i are called **actions** and their conjugate variables θ_i are called **angles** because one usually scales the actions so ordinary phase-space coordinates such as x are 2π periodic in the angles. That is the function on phase space x can be expanded as

$$x(\boldsymbol{\theta}, \mathbf{J}) = \sum_{\mathbf{n}} X_{\mathbf{n}}(\mathbf{J}) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}. \tag{1.6}$$

The Fourier expansion (1.2) from which we started arises by eliminating $\boldsymbol{\theta}$ between equations (1.5) and (1.6).

Whenever the frequencies Ω_i are incommensurable (that is, no relation of the form $\mathbf{n}\cdot\boldsymbol{\Omega} = 0$ exists) the actions constitute a complete set of integrals of motion in the sense that any integral of motion can be obtained as a function of them. Since almost all real numbers are irrational, the frequencies of most orbits are incommensurable and the actions are generically a complete set of integrals.

The DF of an equilibrium mean-field model satisfies $\mathrm{d}f/\mathrm{d}t = 0$, so it is an integral of motion. Consequently, we will assume that at any given time the DF of the mean-field model is a function $f(\mathbf{J}, t)$ of the actions. This assumption is known as **Jeans' theorem**. In a plasma we assume $f(\mathbf{v})$ because in a homogeneous system, $\mathbf{v} = $ constant. Many formulae derived for a plasma will go over to a stellar system with the substitutions $\mathbf{x} \to \boldsymbol{\theta}$, $\mathbf{v} \to \mathbf{J}$.

[1] Binney & Spergel, ApJ, 252, 308 (1982)

[2] Whereas the Fourier decomposition of a periodic function contains only integer multiples of a single fundamental frequency, a quasiperiodic function contains only integer linear combinations of 2 or more fundamental frequencies.

1.2.1 Adiabatic invariance

Action integrals are **adiabatic invariants**: if H evolves on a timescale that is longer than the dynamical time, an orbit of H evolves in such a way that $\mathbf{J}=$ constant. Consequently, when H evolves slowly, the number of particles with $\mathbf{J}$ in each element $\mathrm{d}^3\mathbf{J}$ of action space is unchanged, so the function $f(\mathbf{J})$ is constant.

1.2.2 Hamilton-Jacobi equation

Let $S(\mathbf{x},\mathbf{J})$ be the generating function of the canonical transformation $(\mathbf{x},\mathbf{p}) \leftrightarrow (\boldsymbol{\theta},\mathbf{J})$. Then $\mathbf{p}=\partial S/\partial\mathbf{x}$ and we use this relation to eliminate $\mathbf{p}$ from the statement that the Hamiltonian is constant along the orbit:

$$H\left(\mathbf{x},\frac{\partial S}{\partial \mathbf{x}}\right)=E. \tag{1.7}$$

This **Hamilton-Jacobi equation**, which holds at all points that can be reached by the orbit, is a p.d.e. for S. In practice it can only be solved when the substitution $S(\mathbf{x},\mathbf{J})=\sum_i S_i(x_i,\mathbf{J})$ leads to a clean separation of variables. For example, for planar motion in an axisymmetric $\Phi(r)$ we have

$$H(r,\phi,p_r,p_\phi)=\tfrac{1}{2}p_r^2+\frac{p_\phi^2}{2r^2}+\Phi(r), \tag{1.8}$$

so $2r^2$ times the H-J eqn yields

$$r^2\left(\frac{\partial S_r}{\partial r}\right)^2+2r^2(\Phi-E)=-\left(\frac{\partial S_\phi}{\partial\phi}\right)^2=-L^2, \tag{1.9}$$

where $-L^2$ is a constant of separation. $S_\phi=L\phi$ follows trivially, and almost as easily we get

$$S_r=\int^r \mathrm{d}r\,\sqrt{2(E-\Phi)-\frac{L^2}{r^2}}. \tag{1.10}$$

These operations yield a function $S(\mathbf{x},E,L)$, which is not of the required form: the integrals of motion E,L are (unknown) functions of the required action integrals; the pair (E,L) cannot be complemented by variables to form a set of canonical coordinates. The actions J_i are defined by

$$J_i=\frac{1}{2\pi}\oint_{\gamma_i}\mathrm{d}\mathbf{x}\cdot\mathbf{p}, \tag{1.11}$$

where each path γ_i around the part of phase space accessible to the orbit cannot be deformed into another of the γ_i without leaving the accessible region. (The integrals (1.11) are unchanged by sliding γ_i over the accessible region because the latter has vanishing Poincaré invariant $\sum_i \mathrm{d}x_i\mathrm{d}p_i$.) In the case of 2d motion, we hold r constant along one path, and ϕ constant along the other path. Then the first path trivially yields $J_\phi=L$ and the second path yields

$$J_r(E,L)=\frac{1}{2\pi}\oint\mathrm{d}r\,p_r=\frac{1}{\pi}\int_{r_{\min}}^{r_{\max}}\mathrm{d}r\,\sqrt{2(E-\Phi)-\frac{L^2}{r^2}}. \tag{1.12}$$

To find the angle variables we have to use the chain rule

$$\theta_i=\frac{\partial S_r}{\partial J_i}+\frac{\partial S_\phi}{\partial J_i}=\frac{\partial S_r}{\partial E}\frac{\partial E}{\partial J_i}+\frac{\partial S_r}{\partial L}\frac{\partial L}{\partial J_i}+\frac{\partial S_\phi}{\partial L}\frac{\partial L}{\partial J_i}. \tag{1.13}$$

1.2.3 Choice of actions

The **action integrals** J_i are defined up to a set of discrete canonical transformations (generating function $S(\boldsymbol{\theta},\mathbf{J}')=\boldsymbol{\theta}\cdot\mathbf{M}\cdot\mathbf{J}'$ where the matrix $\mathbf{M}$ has integer elements). For an axisymmetric system the actions are uniquely defined by requiring that

$$\begin{aligned}&J_r \text{ quantifies radial excursions}\\&J_\phi=L_z \text{ is angular momentum about the symmetry axis}\\&J_z \text{ quantifies oscilations perpendicular to the equatorial plane.}\end{aligned} \tag{1.14}$$

In the spherical limit $J_z=L-|L_z|$ is the angular momentum in the (x,y) plane.

1.3 Self-consistent, mean-field model

A stellar system's distribution function (DF) $f(\mathbf{x},\mathbf{v})$ specifies the mass $\mathrm{d}m = f\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{v}$ in each infinitesimal volume of phase space. If the system is in a statistically steady state, Jeans' theorem tells us that f can depend on $(\mathbf{x},\mathbf{v})$ only through $\mathbf{J}(\mathbf{x},\mathbf{v})$, so it can be expressed as a function $f(\mathbf{J})$. In fact any non-negative function of three variables $0 \le J_r < \infty$, $-\infty < J_\phi < \infty$ and $0 \le J_z < \infty$ specifies an axisymmetric stellar system, and a really powerful way of generating models that can be fitted to observational data is simply to write down a likely function.[3]

Given some function $f(\mathbf{J})$ how do we discover what the system looks like in real space?

1) Make a guess $\rho_0(\mathbf{x})$ at its density distribution. The guess doesn't have to be a good one, but you should ensure that its mass satisfies

$$M \equiv (2\pi)^3 \int \mathrm{d}^3\mathbf{J}\, f(\mathbf{J}) = \int \mathrm{d}^3\mathbf{x}\,\rho_0. \tag{1.15}$$

2) Solve Poisson's equation for the potential $\Phi_0(\mathbf{x})$ generated by ρ_0.
3) Obtain the angle-action coordinates for $\rho_0(\mathbf{x})$ and use them to determine a new density distribution

$$\rho_1(\mathbf{x}) = \int \mathrm{d}^3\mathbf{v}\, f[J(\mathbf{x},\mathbf{v})]. \tag{1.16}$$

4) Return to step (2) with ρ_0 replaced by ρ_1 and iterate until $\rho_n(\mathbf{x})$ differs negligibly from ρ_{n-1}. This typically requires ~ 5 iterations.[4]

The only tricky part of this procedure is obtaining the angle-action coordinates of Φ_n. In practice approximations to the true $(\boldsymbol{\theta},\mathbf{J})$ coordinates are used.[5]

1.4 Biorthogonal potential-density pairs

Unfortunately, while Φ is a function of only $\mathbf{x}$, it becomes a function of both $\boldsymbol{\theta}$ and $\mathbf{J}$. So while angle-action variables make dynamics trivial (advance $\boldsymbol{\theta}$ linearly in t), they seriously complicate the solution of Poisson's eqn.

We finesse this difficulty by introducing a basis of **biorthogonal potential-density** pairs. That is, a set of pairs $(\rho^{(\alpha)},\Phi^{(\alpha)})$ such that

$$4\pi G\rho^{(\alpha)} = \nabla^2\Phi^{(\alpha)} \quad \text{and} \quad \int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}\rho^{(\alpha')} = -\mathcal{E}\delta_{\alpha\alpha'}, \tag{1.17}$$

where $\mathcal{E}$ is an arbitrary constant with the dimensions of energy. Given a density distribution $\rho(\mathbf{x})$, we expand it in the basis

$$\rho(\mathbf{x}) = \sum_\alpha A_\alpha \rho^{(\alpha)}(\mathbf{x}) \quad \Rightarrow \quad \begin{cases} \Phi(\mathbf{x}) = \sum_\alpha A_\alpha \Phi^{(\alpha)}(\mathbf{x}), \\ A_\alpha = -\dfrac{1}{\mathcal{E}}\displaystyle\int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}(\mathbf{x})\rho(\mathbf{x}). \end{cases} \tag{1.18}$$

If ρ and Φ are time-dependent, the A_α become time-dependent.

In practice potential-density pairs are complex because they are based on the spherical harmonics Y_l^m – see §2.8 of BT08 for more information. However, we could regard the real and imaginary parts of $Y_l^m(\theta,\phi) = p_l^m(\cos\theta)(\cos\phi + \mathrm{i}\sin\phi)$ as (real) basis functions in their own right. Below, we will find it useful to assume that we are in fact working with real basis functions.

[3] In the 20th c. composers appeared who argued that any series of notes constitutes music. We disagree: writing music involves observing rules regarding scales, chords, etc. Similarly, creating plausible stellar systems requires adherence to rules regarding how $f(\mathbf{J})$ behaves in certain parts of action space. But these rules are a matter of good taste.

[4] Binney, MNRAS, **440**, 787 (2014)

[5] Sanders & Binney, MNRAS, **457**, 2107 (2016)

1.5 Perturbing the DF

Now that we've introduced the idea of a mean-field model and its custom coordinates (angle-action coords), let's obtain the equations that govern small disturbances in f. The basic equation is the Liouville (Vlasov) equation expressing constancy of f along orbits:

$$\begin{aligned} 0 = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} &= \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} \\ &= \frac{\partial f}{\partial t} + [f, H], \end{aligned} \tag{1.19}$$

where $H = \frac{1}{2}v^2 + \Phi$ is the Hamiltonian and [.] denotes a **Poisson bracket** defined by

$$[f, g] \equiv \sum_i \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial v_i} - \frac{\partial f}{\partial v_i} \frac{\partial g}{\partial x_i} \right).$$

Since the $(\boldsymbol{\theta}, \mathbf{J})$ system, like the $(\mathbf{x}, \mathbf{v})$ one, is canonical and Poisson brackets are invariant under changes of canonical coordinates, we can substitute $\mathbf{x} \to \boldsymbol{\theta}$, $\mathbf{v} \to \mathbf{J}$ in all these formulae if we wish. Jeans' theorem follows from the observation that $H(\mathbf{J})$ is a function of only $\mathbf{J}$, so if f also depends only on $\mathbf{J}$, the Poisson bracket $[f, H]$ vanishes and f becomes time-independent.

Now we write

$$f(\boldsymbol{\theta}, \mathbf{J}, t) = f_0(\mathbf{J}) + f_1(\boldsymbol{\theta}, \mathbf{J}, t) \tag{1.20}$$

and

$$H = H_0(\mathbf{J}) + \Phi_1(\boldsymbol{\theta}, \mathbf{J}, t) \tag{1.21}$$

on the understanding that $f_1 \ll f_0$ and $\Phi_1 \ll H_0$. Then from the bilinearity of the Poisson bracket and equation (1.19), we obtain immediately

$$0 = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \boldsymbol{\theta}} \cdot \frac{\partial H_0}{\partial \mathbf{J}} - \frac{\partial f_0}{\partial \mathbf{J}} \cdot \frac{\partial \Phi_1}{\partial \boldsymbol{\theta}} + \mathrm{O}(f_1^2). \tag{1.22}$$

From (1.5) we identify $\partial H_0 / \partial \mathbf{J} = \boldsymbol{\Omega}_0(\mathbf{J})$ as the frequency vector of the unperturbed orbit $\mathbf{J}$ Moreover, incrementing any angle coordinate by 2π brings us back to the same point in phase space (eq. 1.6), so all functions of $\boldsymbol{\theta}$ can be expressed as Fourier series:

$$\begin{aligned} f_1(\boldsymbol{\theta}, \mathbf{J}, t) &= \sum_{\mathbf{n}} \hat{f}_1(\mathbf{n}, \mathbf{J}, t) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} & & & \hat{f}_1(\mathbf{n}, \mathbf{J}, t) &= \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3} f_1(\boldsymbol{\theta}, \mathbf{J}, t) \mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} \\ & & \leftrightarrow & & & \\ \Phi_1(\boldsymbol{\theta}, \mathbf{J}, t) &= \sum_{\mathbf{n}} \hat{\Phi}_1(\mathbf{n}, \mathbf{J}, t) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} & & & \hat{\Phi}_1(\mathbf{n}, \mathbf{J}, t) &= \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3} \Phi_1(\boldsymbol{\theta}, \mathbf{J}, t) \mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}, \end{aligned} \tag{1.23}$$

Using these results, we can rewrite the linearised Vlasov equation (1.22) as

$$0 = \sum_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} \left(\frac{\partial \hat{f}_1}{\partial t} + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0 \hat{f}_1 - \mathrm{i}\mathbf{n} \cdot \frac{\partial \hat{f}_0}{\partial \mathbf{J}} \hat{\Phi}_1 \right). \tag{1.24}$$

Since $\boldsymbol{\theta}$ is arbitrary, for this equation to hold, every coefficient of $\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ must separately vanish, so we obtain an infinite set of equations

$$\frac{\partial \hat{f}_1}{\partial t} = \mathrm{i}\mathbf{n} \cdot \frac{\partial \hat{f}_0}{\partial \mathbf{J}} \hat{\Phi}_1 - \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0 \hat{f}_1 \quad \text{for } \mathbf{n} \text{ with integer components.} \tag{1.25}$$

We use Laplace transforms to solve (1.25): multiplying by e^{-pt} (with $\Re(p) > 0$) and integrating over t, we get[6]

$$p\widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) - \hat{f}_1(\mathbf{n}, \mathbf{J}, 0) + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0 \widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) - \mathrm{i}\mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \widetilde{\Phi}_1(\mathbf{n}, \mathbf{J}, p) = 0, \tag{1.26}$$

[6] While the dimensions of a quantity are unchanged by a hat, a tilde raises the dimensions by a factor T.

where the tildes denote Laplace transforms:

$$\widetilde{f}_1(\mathbf{n},\mathbf{J},p) \equiv \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt}\hat{f}_1(\mathbf{n},\mathbf{J},t). \tag{1.27}$$

Solving for $\widetilde{f}_1$ we have (cf. Schekochihin eqn. 3.8)

$$\widetilde{f}_1(\mathbf{n},\mathbf{J},p) = \frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p) + \hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}. \tag{1.28}$$

This equation provides one connection between a perturbation to the potential Φ_1 and the response f_1 it induces dynamically.

We now need to put into maths the principle that Φ_1 is the potential generated by the perturbation to the density that's associated with f_1. To obtain the coefficients A_α of the potential-density expansion (1.18) of the perturbed density (or at any rate their temporal Laplace transforms), we multiply the left side of (1.28) by $\Phi^{(\alpha)*}\sum_\mathbf{n}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ and integrate over phase space:

$$\begin{aligned}\int \mathrm{d}^3\boldsymbol{\theta}\mathrm{d}^3\mathbf{J}\,\Phi^{(\alpha)*}(\mathbf{x})\sum_\mathbf{n}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\widetilde{f}_1(\mathbf{n},\mathbf{J},p) &= \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}\,\Phi^{(\alpha)*}(\mathbf{x})\widetilde{f}_1(\mathbf{x},\mathbf{v},p)\\ &= \int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}(\mathbf{x})\widetilde{\rho}_1(\mathbf{x},p) = -\mathcal{E}\widetilde{A}_\alpha(p).\end{aligned} \tag{1.29}$$

Here we have exploited the fact the Jacobian between any two sets of canonical coordinates is unity, so $\mathrm{d}^3\boldsymbol{\theta}\mathrm{d}^3\mathbf{J} = \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$. Now operating in the same way on the rhs of eqn (1.28) we have

$$\begin{aligned}&\int \mathrm{d}^3\boldsymbol{\theta}\mathrm{d}^3\mathbf{J}\sum_\mathbf{n}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\Phi^{(\alpha)*}(\mathbf{x})\frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p) + \hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}\\ &\quad= (2\pi)^3\int \mathrm{d}^3\mathbf{J}\sum_\mathbf{n}[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\sum_{\alpha'}\widetilde{A}_{\alpha'}(p)\hat{\Phi}^{(\alpha')}(\mathbf{n},\mathbf{J}) + \hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}.\end{aligned} \tag{1.30}$$

Uniting the two sides (1.29) and (1.30) of equation (1.28) we obtain an equation for $\widetilde{A}_\alpha$:

$$\widetilde{A}_\alpha(p) = -\frac{(2\pi)^3}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_\mathbf{n}\frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\sum_{\alpha'}\widetilde{A}_{\alpha'}(p)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\hat{\Phi}^{(\alpha')}(\mathbf{n},\mathbf{J}) + \hat{f}_1(\mathbf{n},\mathbf{J},0)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}. \tag{1.31}$$

We move the term on the right containing $A_{\alpha'}$ to the left side so we can write

$$\sum_{\alpha'}\epsilon_{\alpha\alpha'}(p)\widetilde{A}_{\alpha'}(p) = -\frac{(2\pi)^3}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_\mathbf{n}\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}, \tag{1.32a}$$

where

$$\epsilon_{\alpha\alpha'}(p) \equiv \delta_{\alpha\alpha'} + \frac{(2\pi)^3}{\mathcal{E}}\mathrm{i}\int \mathrm{d}^3\mathbf{J}\sum_\mathbf{n}\frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\hat{\Phi}^{(\alpha')}(\mathbf{n},\mathbf{J}) \tag{1.32b}$$

is the analogue of the dielectric function (cf. Schekochihin eqn. 3.11). In both integrals over $\mathbf{J}$ in equations (1.32) we must use the Landau prescription. That is, we must ensure that $\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0$ passes to the left of p in the complex plane (Box 1.2).

After computing the inverse of the dimensionless matrix $\boldsymbol{\epsilon}$, we have an explicit expression for $\widetilde{A}_\alpha(p)$. Multiplying this by $\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})$ and summing over α we obtain the Laplace transform of the potential perturbation arising from the initial condition $f_1(\mathbf{n},\mathbf{J},0)$:

$$\begin{aligned}\widetilde{\Phi}_1(\mathbf{n}',\mathbf{J}',p) &= \sum_{\alpha'}\widetilde{A}_{\alpha'}(p)\hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}')\\ &= -\frac{(2\pi)^3}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_\mathbf{n}\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}\sum_{\alpha\alpha'}\hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}')\boldsymbol{\epsilon}^{-1}_{\alpha'\alpha}(p)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\\ &= -(2\pi)^3\int \mathrm{d}^3\mathbf{J}\sum_\mathbf{n}E_{\mathbf{n}'\mathbf{n}}(\mathbf{J}',\mathbf{J},p)\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0},\end{aligned} \tag{1.33a}$$

Box 1.1: Stability of a collisionless system

If we recover the temporal dependence of Φ_1 by taking the inverse Laplace transform of equation (1.33a), we obtain a sum of terms with exponential time dependence $e^{p_i t}$ (Schekochihin eqn. 3.16), where p_i is the value of the Laplace transform variable at which the matrix $\mathbf{E}$ has a pole in the sense that it is the inverse of a singular matrix $\boldsymbol{\epsilon}$. Consequently, the stability of a system at the level of collisionless dynamics is determined by whether the dielectric matrix $\boldsymbol{\epsilon}$ is singular at any value p_0 of the Laplace transform variable with $\Re p_0 > 0$. We say that each p_i is associated with a **normal mode** of the system. In a stable system the normal modes are all neutral ($\Re p_i = 0$) or damped ($\Re p_i < 0$).

where

$$E_{\mathbf{n}'\mathbf{n}}(\mathbf{J}', \mathbf{J}, p) \equiv \frac{1}{\mathcal{E}} \sum_{\alpha\alpha'} \hat{\Phi}^{(\alpha')}(\mathbf{n}', \mathbf{J}') \boldsymbol{\epsilon}^{-1}_{\alpha'\alpha}(p) [\hat{\Phi}^{(\alpha)}(\mathbf{n}, \mathbf{J})]^*, \tag{1.33b}$$

has dimensions $M^{-1}L^2T^{-2}$ and is (to within a factor $\mathcal{E}$) $\boldsymbol{\epsilon}^{-1}$ written in the $(\mathbf{n}, \mathbf{J})$ basis rather than the $(\boldsymbol{\alpha}, \mathbf{J})$ basis. Equation (1.33a) is analogous to Schekochihin eqn. (3.13) in giving the Laplace transform of the response potential set up by a specified initial condition. It's more complicated than Schekochihin eqn. (3.13) because: (a) in the latter Poisson's equation is solved by simply dividing by k^2 while here we do acrobatics with the potential basis functions; (b) we have $\mathbf{E}$ where Schekochihin eqn. 3.13 has $1/\epsilon$ and the case $\epsilon = 0$ becomes the case in which our matrix $\boldsymbol{\epsilon}$ has no inverse, so $\mathbf{E}$, which is basically this inverse, diverges; (c) Schekochihin eqn. (3.13) involves an integral over $\mathbf{v}$ with the denominator of the integrand linear in $\mathbf{v}$, while here we integrate over $\mathbf{J}$ and the denominator involves the non-linear function $\mathbf{n} \cdot \boldsymbol{\Omega}_0(\mathbf{J})$. The generalisation of the Landau prescription to this more complex context is given in Box 1.2.

Box 1.2: The Landau prescription with actions

We often encounter, as in eqns (1.32), an integral over action space with a denominator that vanishes if $p = -\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}(\mathbf{J})$. In a plasma, analogous integrals occur with denominator $p+\mathrm{i}\mathbf{k}\cdot\mathbf{v}$ and we evaluate them using the Landau contour. To solve our more complex problem we make a coordinate change from $\mathbf{J} \to (x, y, z)$, where $z \equiv \mathbf{n}\cdot\mathbf{\Omega}$ and (x, y) is a coordinate system for the 2-surfaces $z = \text{const}$. Then

$$\int \mathrm{d}^3\mathbf{J}\,\frac{k(\mathbf{J})}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} = \int_{-\infty}^{\infty} \mathrm{d}z\,\frac{K(z)}{p+\mathrm{i}z}, \tag{1}$$

where

$$K(z) \equiv \int \mathrm{d}x\mathrm{d}y\,\frac{\partial(\mathbf{J})}{\partial(x,y,z)}k(\mathbf{J}).$$

The integral on the right of (1) is now in just the form considered by Landau. We write $p = \gamma - \mathrm{i}\omega$ with $\gamma > 0$, and have

$$\int_{-\infty}^{\infty} \mathrm{d}z\,\frac{K(z)}{p+\mathrm{i}z} = \int \mathrm{d}z\,\frac{K(z)}{\mathrm{i}(z-\omega)+\gamma} = -\mathrm{i}\int \mathrm{d}z\,\frac{K(z)}{z-(\omega+\mathrm{i}\gamma)}.$$

The z contour (real axis) passes under the pole, so in the limit $\gamma \to 0$ this becomes

$$\int_{-\infty}^{\infty} \mathrm{d}z\,\frac{K(z)}{p+\mathrm{i}z} = -\mathrm{i}\left(\mathcal{P}\int \mathrm{d}z\,\frac{K(z)}{z-\omega} + \mathrm{i}\pi K(\omega)\right) = -\mathrm{i}\mathcal{P}\int \mathrm{d}z\,\frac{K(z)}{z-\omega} + \pi K(\omega). \tag{2}$$

Now let's transform $\int \mathrm{d}^3\mathbf{J}\,k(\mathbf{J})\delta(\mathbf{n}\cdot\mathbf{\Omega}-\omega)$ into the (x, y, z) system:

$$\int \mathrm{d}^3\mathbf{J}\,k(\mathbf{J})\delta(\mathbf{n}\cdot\mathbf{\Omega}-\omega) = \int \mathrm{d}z\,K(z)\delta(z-\omega) = K(\omega).$$

When we use this equation in (2), we obtain the needed analogue of the Plemelj formula.

$$\int \mathrm{d}^3\mathbf{J}\,\frac{k(\mathbf{J})}{p+\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} = -\mathrm{i}\mathcal{P}\int \mathrm{d}z\,\frac{K(z)}{z-\omega} + \pi\int \mathrm{d}^3\mathbf{J}\,k(\mathbf{J})\delta(\mathbf{n}\cdot\mathbf{\Omega}-\omega) \qquad (p = -\mathrm{i}\omega + 0). \tag{3}$$

2

Evolution of the mean-field model

We have been studying the properties of mean-field equilibrium systems. Such systems are fully characterised by a non-negative DF of the form $f(\mathbf{J})$. We have described how the system's real-space properties can be extracted from $f(\mathbf{J})$ and how to compute the evolution of the DF when at $t = 0$ it differs very slightly from $f(\mathbf{J})$. In all the above we have been imagining that the system comprises an extremely large number of particles with extremely low masses, so statistical fluctuations of the density around its mean value, $\rho(\mathbf{x}) = \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v})$, vanish. In this section we explore how to compute the evolution of f that occurs because its constuent particles have non-zero masses, so ρ and Φ fluctuate around their mean values.

Recall from Paul Dellar's discussion of the BBGKY hierarchy that the 1-particle DF $f(\mathbf{x}, \mathbf{v})$ satisfies a Boltzmann equation in which the 2-particle correlation function $g^{(2)}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}')$ appears (Problem 7):

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{\mathbf{w}} = (N-1)\int \mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{v}'\,\frac{\partial u(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}'}\cdot\frac{\partial g^{(2)}(\mathbf{w}, \mathbf{w}')}{\partial \mathbf{v}}, \tag{2.1}$$

where $\mathbf{w} \equiv (\mathbf{x}, \mathbf{v})$ denotes position in phase space and $u(\mathbf{x} - \mathbf{x}')$ is the interaction potential between two particles. The physical content of this equation is that evolution of the mean-field model, $f(\mathbf{x}, \mathbf{v})$, is driven by the tendency, encoded in $g^{(2)}$ for particles to cluster together, so you are more likely to find a second particle near you if you stand on a particle than if you stand in a random location. Heyvaerts[1] obtains from equation (2.1) the equation for the evolution of f, which is what we seek in this section, but we'll proceed along a different path, similar to that laid out by Chavanis.[2]

We argue that the small-scale structure is unimportant so we should be able to compute everything in terms of a smooth potential $\Phi(\mathbf{x}, t)$ providing we properly account for the fluctuations in Φ.

We divide the actual DF f into that of a slowly evolving mean-field model f_0 and a fluctuating part f_1. We similarly divide the potential into mean-field and fluctuating parts $\Phi_0(\mathbf{x}, t)$ and $\Phi_1(\mathbf{x}, t)$ with corresponding Hamiltonians H_0 and $H_1 = \Phi_1$. Here the key is that f_0 and Φ_0 change only over many dynamical times, while f_1 and Φ_1 change significantly on a dynamical time. Since f evolves in the smooth potential Φ, it satisfies the CBE. We write the latter as

$$\begin{aligned} 0 = \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial(f_0+f_1)}{\partial t} + [f_0+f_1, H_0+H_1] \\ &= \frac{\partial f_0}{\partial t} + [f_0, H_0] + [f_1, H_1] + \frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, H_1]. \end{aligned} \tag{2.2}$$

[1] J. Heyvaerts, MNRAS, 407, 355 (2010)
[2] P.-H. Chavanis, Physica A, 391, 3680 (2012).

When we take the expectation[3] of this equation, the last three terms vanish because they are linear in the fluctuations, and we get an equation for the evolution of the mean-field model

$$\frac{\partial f_0}{\partial t} + [f_0, H_0] = -\langle [f_1, H_1] \rangle \,. \tag{2.3a}$$

Subtracting this from (2.2) we get an equation for the dynamical evolution of the fluctuations

$$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, H_1] = 0, \tag{2.3b}$$

which is essentially the same as equation (1.22). Our the strategy is to use (2.3b) to compute the expectation value on the right of (2.3a).

2.1 Dynamics of fluctuations

In the right side of diffusion equation (2.3a), we replace f_1 and Φ_1 by their Fourier expansions in $\boldsymbol{\theta}$ (eq. 1.23). We also take advantage of the expectation operator $\langle . \rangle$ to integrate over all angles. Then the right side of equation (2.3a) becomes

$$\begin{aligned}\langle [f_1, \Phi_1] \rangle = \Bigg\langle \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3} \Bigg(&\sum_{\mathbf{n}} \hat{f}_1(\mathbf{n}, \mathbf{J}, t) \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} \mathrm{i}\mathbf{n} \cdot \sum_{\mathbf{n}'} \frac{\partial \hat{\Phi}_1(\mathbf{n}', \mathbf{J}, t)}{\partial \mathbf{J}} \mathrm{e}^{\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\theta}} \\ &- \sum_{\mathbf{n}} \frac{\partial \hat{f}_1(\mathbf{n}, \mathbf{J}, t)}{\partial \mathbf{J}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} \cdot \sum_{\mathbf{n}'} \mathrm{i}\mathbf{n}' \hat{\Phi}_1(\mathbf{n}', \mathbf{J}, t) \mathrm{e}^{\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\theta}} \Bigg) \Bigg\rangle \\ = \mathrm{i} \frac{\partial}{\partial \mathbf{J}} \cdot \Bigg\langle &\sum_{\mathbf{n}} \mathbf{n} \hat{f}_1(\mathbf{n}, \mathbf{J}, t) \hat{\Phi}_1(-\mathbf{n}, \mathbf{J}, t) \Bigg\rangle .\end{aligned} \tag{2.4}$$

Hence the equation for the evolution of the mean-field model is

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}, \tag{2.5a}$$

where the flux of stars in action space[4] is

$$\mathbf{F} = \mathrm{i} \Bigg\langle \sum_{\mathbf{n}} \mathbf{n} \hat{f}_1(\mathbf{n}, \mathbf{J}, t) \hat{\Phi}_1(-\mathbf{n}, \mathbf{J}, t) \Bigg\rangle . \tag{2.5b}$$

The divergence on the right of (2.5a) guarantees conservation of stars.

Rewriting (2.5b) in terms of Laplace transforms, it becomes

$$\mathbf{F}(\mathbf{J}) = \mathrm{i} \Bigg\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \widetilde{\Phi}_1(-\mathbf{n}, \mathbf{J}, p') \Bigg\rangle . \tag{2.6}$$

Now we use equation (1.28) to eliminate $\widetilde{f}_1$

$$\mathbf{F}(\mathbf{J}) = \mathrm{i} \Bigg\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \left(\frac{\mathrm{i}\mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \widetilde{\Phi}_1(\mathbf{n}, \mathbf{J}, p) + \hat{f}_1(\mathbf{n}, \mathbf{J}, 0)}{p + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0} \right) \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \widetilde{\Phi}_1(-\mathbf{n}, \mathbf{J}, p') \Bigg\rangle . \tag{2.7}$$

This expression for the diffusive flux is made up of a part that's proportional to $\left\langle \widetilde{\Phi}_1(\mathbf{n}) \widetilde{\Phi}_1(-\mathbf{n}) \right\rangle$ that will be non-vanishing regardless of the physical cause of fluctuations in the potential, and a part $\left\langle \widetilde{f}_1(\mathbf{n}) \widetilde{\Phi}_1(-\mathbf{n}) \right\rangle$ that will be non-vanishing only to the extent that the fluctuations in Φ

[3] The expectation can be considered an averaging over different statistically equivalent initial conditions or an average over a time long compared to the lifetime of a fluctuation.

[4] Strictly, the density of stars in action space is $(2\pi)^3 f_0(\mathbf{J})$ and the action-space flux is $(2\pi)^3 \mathbf{F}(\mathbf{J})$ rather than $\mathbf{F}(\mathbf{J})$, but in heuristic discussions it's convenient to ignore the factor $(2\pi)^3$.

are generated by the fluctuations in f. Moreover, the first term is proportional to the gradient of $f_0(\mathbf{J})$ while the second is not. These distinctions will prove important (e.g., Problem 10), so we explicitly break $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ into two parts,

$$\mathbf{F}_1 = \mathrm{i}\sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \frac{\left\langle \hat{f}_1(\mathbf{n},\mathbf{J},0)\widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p') \right\rangle}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}$$
$$\mathbf{F}_2 = -\sum_{\mathbf{n}} \mathbf{n}\,\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \frac{\left\langle \widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p)\widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p') \right\rangle}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0}.$$

Using (1.33a) to eliminate $\widetilde{\Phi}_1$, these fluxes become

$$\mathbf{F}_1(\mathbf{J}) \equiv -(2\pi)^3\mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0} \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{-\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p') \frac{\hat{f}_1(\mathbf{n}',\mathbf{J}',0)}{p' + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'} \right\rangle$$
$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) \equiv (2\pi)^6\mathrm{i}\Bigg\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p) \frac{\hat{f}_1(\mathbf{n}',\mathbf{J}',0)}{p + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'} \\ \times \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \int \mathrm{d}^3\mathbf{J}'' \sum_{\mathbf{n}''} E_{-\mathbf{n}\mathbf{n}''}(\mathbf{J},\mathbf{J}'',p') \frac{\hat{f}_1(\mathbf{n}'',\mathbf{J}'',0)}{p' + \mathrm{i}\mathbf{n}''\cdot\mathbf{\Omega}_0''} \Bigg\rangle.\end{aligned} \tag{2.8}$$

The expectation-value brackets $\langle . \rangle$ imply that we require the expectation $\hat{f}_1(\mathbf{n},\mathbf{J},0)\hat{f}_1(\mathbf{n}',\mathbf{J}',0)$ of the initial conditions. In Box 2.1 we show that

$$\left\langle \hat{f}_1(\mathbf{n},\mathbf{J},0)\hat{f}_1(\mathbf{n}',\mathbf{J}',0) \right\rangle = \frac{1}{(2\pi)^3}\delta_{\mathbf{n},-\mathbf{n}'}\delta(\mathbf{J}-\mathbf{J}')mf_0(\mathbf{J}). \tag{2.9}$$

Inserting this and carrying out the integral over $\mathbf{J}'$ in the equation for $\mathbf{F}_1$ and over $\mathbf{J}''$ in the equation for $\mathbf{F}_2$, we get

$$\begin{aligned}\mathbf{F}_1(\mathbf{J}) &= -\mathrm{i}m\sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{1}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0} \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},p') \frac{f_0(\mathbf{J})}{p' - \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0} \\ \mathbf{F}_2(\mathbf{J}) &= -(2\pi)^3 m \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p) \frac{1}{p + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'} \\ &\quad \times \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',p') \frac{f_0(\mathbf{J}')}{p' - \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'}.\end{aligned} \tag{2.10}$$

The expression for $\mathbf{F}_1$ is easy to simplify further because $E_{-\mathbf{n}-\mathbf{n}}$ won't contribute a pole at $\Re(p') \geq 0$: if it had such a pole, the underlying model would be unstable (Box 1.1), and we are interested in the case when it's stable. So the only singularity we need consider is the obvious one when $p' = \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0$. Similarly, the integration over p follows immediately from the pole at $p = -\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0$. So we have

$$\mathbf{F}_1(\mathbf{J}) = -\mathrm{i}m\sum_{\mathbf{n}} \mathbf{n}E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)f_0(\mathbf{J}). \tag{2.11}$$

Notice that the time dependencies introduced by the two inverse Laplace transforms have cancelled, so the flux $\mathbf{F}_1$ is constant.

Now we turn to $\mathbf{F}_2$. The integral over p' is straightforward because the integrand has only the obvious pole at $p' = \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}'$. After doing the p' integral we have

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -(2\pi)^3 m \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0} \\ \times \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} \mathrm{e}^{\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0' t} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p) E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0') \frac{f_0(\mathbf{J}')}{p + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'}\end{aligned} \tag{2.12}$$

Box 2.1: Expectation value of the initial conditions

We require $\left\langle \hat{f}_1(\mathbf{n},\mathbf{J},0)\hat{f}_1(\mathbf{n}',\mathbf{J}',0)\right\rangle$. We drop the time slot for brevity and recall that f_1 is the difference between the actual DF and the mean-field model, which has DF $f_0(\mathbf{J})$. The actual DF is a sum of one delta-function for each particle:

$$f(\boldsymbol{\theta},\mathbf{J}) = m\sum_i \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i).$$

Thus bearing in mind that $\langle f(\boldsymbol{\theta},\mathbf{J})\rangle = f_0(\mathbf{J})$,

$$\begin{aligned}\big\langle f_1(\boldsymbol{\theta},\mathbf{J})f_1(\boldsymbol{\theta}',\mathbf{J}')\big\rangle &= \big\langle \big(f(\boldsymbol{\theta},\mathbf{J})-f_0(\mathbf{J})\big)\big(f(\boldsymbol{\theta}',\mathbf{J}')-f_0(\mathbf{J}')\big)\big\rangle\\ &= \big\langle f(\boldsymbol{\theta},\mathbf{J})f(\boldsymbol{\theta}',\mathbf{J}')\big\rangle - f_0(\mathbf{J})f_0(\mathbf{J}')\\ &= m^2\sum_{ij}\big\langle \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\big\rangle - f_0(\mathbf{J})f_0(\mathbf{J}').\end{aligned}$$

Now

$$\begin{aligned}\sum_{ij}\big\langle \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\big\rangle &= \sum_{i\neq j}\big\langle \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\big\rangle\\ &\quad+\sum_i\big\langle \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\mathbf{J}'-\mathbf{J})\big\rangle\\ &= m^{-2}f_0(\mathbf{J})f_0(\mathbf{J}') + m^{-1}f_0(\mathbf{J})\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\mathbf{J}-\mathbf{J}'),\end{aligned}$$

where we have assumed that the particles are uniformly distributed in $\boldsymbol{\theta}$ and uncorrelated (so the expectation value of products of delta-functions associated with different particles is the product of the expectation values of the individual terms). When the last equation is used in the previous equation, we obtain

$$\big\langle f_1(\boldsymbol{\theta},\mathbf{J})f_1(\boldsymbol{\theta}',\mathbf{J}')\big\rangle = mf_0(\mathbf{J})\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\mathbf{J}-\mathbf{J}'),$$

which simply states that particles are only correlated with themselves. Finally Fourier transforming

$$\begin{aligned}\left\langle \hat{f}_1(\mathbf{n},\mathbf{J})\hat{f}_1(\mathbf{n}',\mathbf{J}')\right\rangle &= mf_0(\mathbf{J})\delta(\mathbf{J}-\mathbf{J}')\int\frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{\theta}'}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}(\mathbf{n}\cdot\boldsymbol{\theta}+\mathbf{n}'\cdot\boldsymbol{\theta}')}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\\ &= (2\pi)^{-3}mf_0(\mathbf{J})\delta(\mathbf{J}-\mathbf{J}')\delta_{\mathbf{n},-\mathbf{n}'}.\end{aligned}$$

Now we perform the integral over $\mathbf{J}'$ using the Landau prescription (Box 1.2) to handle the pole at $\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\Omega}_0' = -p$:

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -(2\pi)^3m\sum_{\mathbf{n}}\mathbf{n}\int\frac{\mathrm{d}p}{2\pi\mathrm{i}}\mathrm{e}^{pt}\frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0}\bigg(-\mathrm{i}P+\pi\int\mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathrm{e}^{-pt}\delta(\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\Omega}_0'-\mathrm{i}p)\\ \times E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p)E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',-p)f_0(\mathbf{J}')\bigg),\end{aligned}\tag{2.13}$$

where P is the (real) principal part of the integral. It is now straightforward to execute the integral over p because the integrand has just the simple pole at $p=-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}$. After integration over p we have

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -(2\pi)^3m\sum_{\mathbf{n}}\mathbf{n}\mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0t}\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\bigg(-\mathrm{i}P+\pi\int\mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0t}\delta(\mathbf{n}'\cdot\boldsymbol{\Omega}_0'-\mathbf{n}\cdot\boldsymbol{\Omega}_0)\\ \times E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0)E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0)f_0(\mathbf{J}')\bigg),\end{aligned}\tag{2.14}$$

We now argue that since $\mathbf{F}_2$ is real, the contribution from the principal part, P, must vanish, and we have finally

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -\tfrac{1}{2}(2\pi)^4 m \sum_{\mathbf{n}} \mathbf{n}\Big(\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\Big) \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} \delta(\mathbf{n}'\cdot\boldsymbol{\Omega}'_0 - \mathbf{n}\cdot\boldsymbol{\Omega}_0) \\ \times E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0) E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0) f_0(\mathbf{J}').\end{aligned} \tag{2.15}$$

Notice that the time dependence has disappeared from $\mathbf{F}_2$ as it did from $\mathbf{F}_1$.

At this point we assume that we are working with real basis functions $\Phi^{(\alpha)}$ for then by the bottom-right equation of (1.23), $[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^* = \hat{\Phi}^{(\alpha)}(-\mathbf{n},\mathbf{J})$. Also $[\boldsymbol{\epsilon}(p)]^* = \boldsymbol{\epsilon}(p^*)$ (Problem 8). Consequently, from (1.33b)

$$\begin{aligned}[E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega})]^* &= \frac{1}{\mathcal{E}}\sum_{\alpha\alpha'}[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*[\boldsymbol{\epsilon}^{-1}_{\alpha\alpha'}(-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega})]^*\hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}') \\ &= \frac{1}{\mathcal{E}}\sum_{\alpha\alpha'}\hat{\Phi}^{(\alpha)}(-\mathbf{n},\mathbf{J})\boldsymbol{\epsilon}^{-1}_{\alpha\alpha'}(\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega})[\hat{\Phi}^{(\alpha')}(-\mathbf{n}',\mathbf{J}')]^* \\ &= E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}).\end{aligned} \tag{2.16}$$

Consequently, our expression (2.15) can be simplified to

$$\mathbf{F}_2(\mathbf{J}) = -\tfrac{1}{2}(2\pi)^4 m \sum_{\mathbf{n}\mathbf{n}'} \mathbf{n}\big(\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\big) \int \mathrm{d}^3\mathbf{J}' \big|E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0)\big|^2 f_0(\mathbf{J}')\delta(\mathbf{n}'\cdot\boldsymbol{\Omega}'_0 - \mathbf{n}\cdot\boldsymbol{\Omega}_0). \tag{2.17}$$

This completes our computation of the diffusive flux in action space that's engendered by Poisson fluctuations in the density:

$$\begin{aligned}\mathbf{F}(\mathbf{J}) &= \mathbf{F}_1(\mathbf{J}) + \mathbf{F}_2(\mathbf{J}) \\ &= -\mathbf{D}_1(\mathbf{J}) f_0 - \mathbf{D}_2(\mathbf{J})\cdot\frac{\partial f_0}{\partial \mathbf{J}},\end{aligned} \tag{2.18}$$

where $\mathbf{D}_1$ is the (vector) **drag coefficient** and $\mathbf{D}_2$ is the (tensor) **diffusion coefficient**:

$$\begin{aligned}\mathbf{D}_1(\mathbf{J}) &= \mathrm{i}m\sum_{\mathbf{n}} E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0)\,\mathbf{n} \\ \mathbf{D}_2(\mathbf{J}) &= \tfrac{1}{2}(2\pi)^4 m\sum_{\mathbf{n}\mathbf{n}'}\int \mathrm{d}^3\mathbf{J}'\big|E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0)\big|^2 f_0(\mathbf{J}')\delta(\mathbf{n}'\cdot\boldsymbol{\Omega}'_0 - \mathbf{n}\cdot\boldsymbol{\Omega}_0)\,\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{2.19}$$

Notice that the sign of $\mathbf{D}_2$ is positive, so the flux that it generates is in the opposite direction to the gradient of f_0: stars diffuse away from regions of high phase-space density. Whereas the flux of heat in a metal bar, $\mathbf{q} = -\kappa\nabla T$ is simply proportional to the gradient of the heat-density T, our diffusive flux has, in addition to a term that's proportional to the gradient of the star density, a term that's proportional to the density itself. To understand the necessity of this additional term, consider how the system would evolve if it were absent. Then stars would diffuse from their modest initial actions to ever higher actions, so eventually the density of stars would become uniform throughout phase space, just as heat diffusion will eventually make the temperature uniform throughout a bar. However, energy conservation, which is encoded in the dynamics we have been using, excludes a uniform distribution of stars in action space, since larger actions are associated with more energy. Consequently, the tendency of the term in $\mathbf{F}$ proportional to $\partial f_0/\partial\mathbf{J}$ to drive the system to uniformity in action space has to be counteracted by the term proportional to f_0, which generates a net drift towards the origin of action space.

In thermal equilibrium, $\mathbf{F}$ must vanish by detailed balance. Then the DF $f_0 = \exp(-\beta H)$, where H is the Hamiltonian and $\beta = (k_{\mathrm{B}}T)^{-1}$ is the inverse temperature. Since $\partial H/\partial\mathbf{J} = \boldsymbol{\Omega}_0$, for $\mathbf{F}$ to vanish the diffusion coefficients (which depend on f_0) must satisfy

$$\mathbf{D}_1(\mathbf{J}) - \beta\mathbf{D}_2(\mathbf{J})\cdot\boldsymbol{\Omega}_0(\mathbf{J}) = 0 \tag{2.20}$$

everywhere in action space. This relation provides a useful check on any formulae for the diffusion coefficients (Problem 9). It also suggests that whatever the origin of the fluctuations that drive diffusion (here Poisson fluctuations), $\mathbf{D}_1$ and $\mathbf{D}_2$ will be closely related to one another. In fact, from our expression for $\mathbf{D}_1$ one can derive (Appendix A)

$$\mathbf{D}_1(\mathbf{J}) = -\tfrac{1}{2}(2\pi)^4 m \sum_{\mathbf{n}\mathbf{n}'} \int \mathrm{d}^3\mathbf{J}' \big| E_{\mathbf{n}\mathbf{n}'}(\mathbf{J}, \mathbf{J}', -\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0) \big|^2 \mathbf{n}' \cdot \frac{\partial f_0}{\partial \mathbf{J}'} \delta(\mathbf{n}'\cdot\mathbf{\Omega}_0' - \mathbf{n}\cdot\mathbf{\Omega}_0)\, \mathbf{n}, \qquad (2.21)$$

which is extremely similar to our expression for $\mathbf{D}_2$.

Equation (2.21) for $\mathbf{D}_1$ and our equation (2.19) for $\mathbf{D}_2$ give the $\mathbf{D}_i(\mathbf{J})$ as sums of contributions from stars at any point $\mathbf{J}'$ at which stars "resonate" with stars at $\mathbf{J}$ – two stars resonate in the sense that the $\mathbf{n}'$ harmonic of one star coincides with the $\mathbf{n}$ harmonic of the other. $\mathbf{D}_1$ and $\mathbf{D}_2$ are proportional to the values taken by $\partial f_0/\partial \mathbf{J}'$ and $f_0(\mathbf{J}')$, respectively, because the strength of the oscillating field that's created by the stars at $\mathbf{J}'$ is proportional to the number of stars at $\mathbf{J}'$. On account of the vector $\mathbf{n}$ that occurs in $\mathbf{D}_1$ and the diadic $\mathbf{n}\otimes\mathbf{n}$ in $\mathbf{D}_2$, the diffusion tensor is highly anisotropic in the sense that stars diffuse anomalously fast in the direction $\mathbf{n}$ that yields the largest number of resonant stars.

3

Diffusion in a galactic disc

The formalism developed in the last section gives fascinating insight into the dynamics of galactic discs similar to that in which we reside. These systems were among the first to be studied by N-body simulation when electronic computers became widely available, but it is only recently that we have achieved a reasonable understanding of their dynamics.

Fouvry et al. (arXiv150706887) have applied the formalism of Chapter 2 to razor-thin discs: restricting motion to the xy plane significantly simplifies the computations. First, angle-action coordinates are readily constructed for an axisymmetric disc (Problem 3). Second, Kalnajs (1976) has defined a convenient set of orthonormal potential-density pairs

$$\Phi^\alpha(r,\phi) = \mathrm{e}^{\mathrm{i}l\phi}\Phi^l_n(r) \qquad \rho^\alpha(r,\phi) = \mathrm{e}^{\mathrm{i}l\phi}\rho^l_n(r), \tag{3.1}$$

where $\alpha = (l,n)$. Φ^l_n is a specified polynomial and ρ^l_n is a polynomial in r times a half power of $1 - r^2/r_0^2$, where r_0 is the edge of the disc.

They considered a disc that is confined by a potential that generates a circular speed $v_\mathrm{c} = (R\partial\Phi/\partial R)^{1/2}$ that is everywhere constant. If the disc generated this potential on its own, its surface density $\Sigma(R)$ would be proportional to R^{-2}. It is more realistic (and numerically more convenient) to assume that Φ is generated by three components: (i) a bulge that dominates the mass density near the origin, (ii) a dark halo that dominates the mass density far from the centre, and (iii) the disc, which contributes ~ 0.5 of the radial force at intermediate radii. One says that a "Mestel" disc with $\Sigma(R) \propto R^{-2}$ has been "tapered" at small and large radii to accommodate the bulge and the dark halo. The unperturbed DF is

$$f_0(E,J_\phi) = \xi C J_\phi^q \mathrm{e}^{-E/\sigma_r^2} T_\mathrm{in}(J_\phi) T_\mathrm{out}(J_\phi), \tag{3.2a}$$

where $E = \frac{1}{2}(v_R^2 + v_\phi^2) + \Phi$, C normalises the DF such that with $\xi = T_\mathrm{in} = T_\mathrm{out} = 1$ the disc generates the entire potential, σ_r is a parameter that controls the magnitude of stars' random motions, and

$$q = (v_\mathrm{c}/\sigma_r)^2 - 1 \tag{3.2b}$$

was taken to have the value 11.4. Finally the taper functions are

$$T_\mathrm{in}(J_\phi) = \frac{J_\phi^4}{(R_\mathrm{in}v_\mathrm{c})^4 + J_\phi^4} \qquad T_\mathrm{out}(J_\phi) = \frac{(R_\mathrm{out}v_\mathrm{c})^5}{(R_\mathrm{out}v_\mathrm{c})^5 + |J_\phi|^5}, \tag{3.2c}$$

where $R_\mathrm{out} = 11.5R_\mathrm{in}$. By increasing ξ between zero and unity, the dynamical importance of the disc's self-gravity can be increased from unimportant to dominant. For $\xi \simeq 0.5$ this disc is known to be stable in the sense (Box 1.1) that all its normal modes are damped (Toomre 1981).

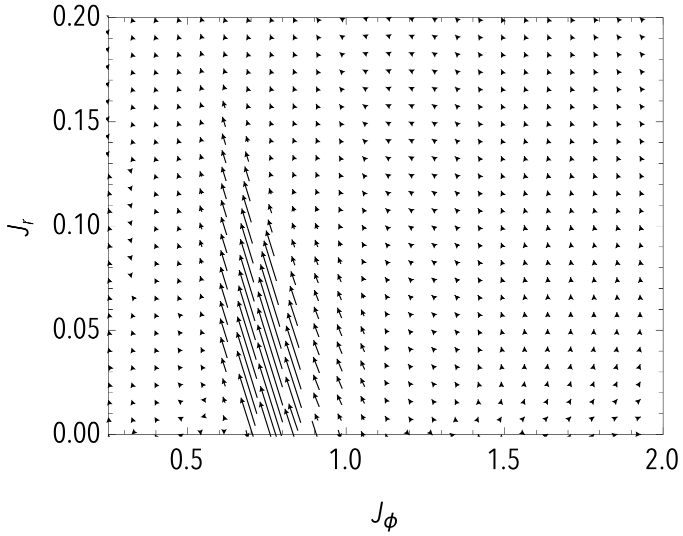

Figure 3.1 The diffusive flux of stars in action space for a tapered Mestel disc with active mass fraction $\xi = 0.5$ (From Fouvry et al 2015).

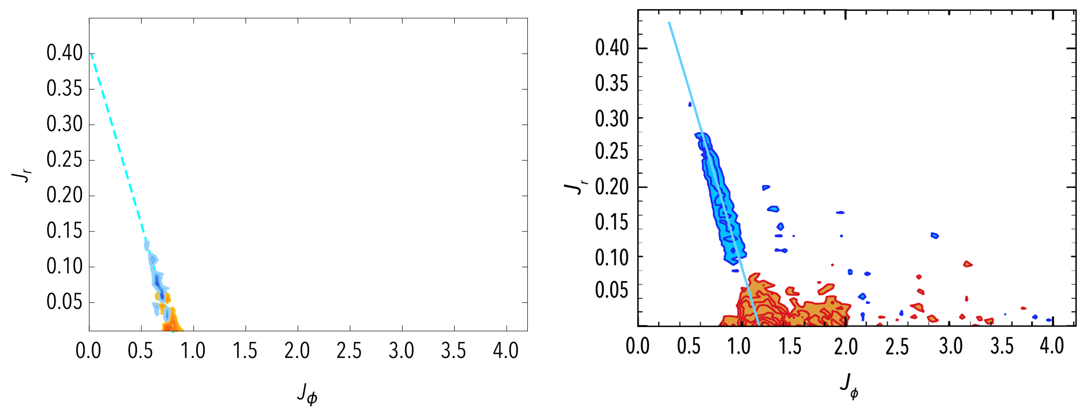

Figure 3.2 Left panel: a plot of div**F**, computed from equation (A.7), with blue indicating negative and red positive values. Right panel: a corresponding plot of the increment (blue) or decrement (red) in the DF during an N-body simulation of the same disc by Sellwood (2012).

In Figure 3.1 arrows show the diffusive flux **F** computed from equation (A.7). We see that **F** is small except along a ridge that slopes leftwards up from the J_ϕ axis (which is where the stars of a cool disc are strongly concentrated). The narrowness of this ridge is emphasised by the left panel of Figure 3.2, which shows div**F**. We see that div**F** is negligible except along a narrow ridge, where it is positive lower down and negative higher up, indicating that stars are diffusing from near circular orbits to more eccentric orbits with slightly less angular momentum. The ridge of non-negligible **F** nearly coincides with a line

$$2\Omega_\phi - \Omega_r = \text{constant} \equiv 2\omega_\text{p}. \tag{3.3}$$

Stars on this line are said to be at the **inner Lindblad resonance** of a perturbation. This perturbation is a bi-symmetric structure that rotates in the same sense as the stars in the disc with angular velocity ω_p. So Figure 3.1 indicates that the dynamics of the disc are dominated by a coherent structure. Given that this is a stable disc, how can this be?

The answer is that although all the disc's normal modes are damped, some are only weakly damped. Consequently, for some values of p the matrix $\mathbf{E}$ that occurs squared in equation (A.7) becomes large. Consequently $\mathbf{F}$ becomes large at points in action space at which i$\mathbf{n}\cdot\mathbf{\Omega}$ coincides (for some $\mathbf{n}$) with such a value of p.

Actually, to get a large value of $\mathbf{F}$ it is not sufficient to have a large value of $|\mathbf{E}|^2$; the product $(\partial f_0/\partial \mathbf{J}) f_0(\mathbf{J}')$ should also be large. In short, the requirements for obtaining a large flux are quite specific and it is perhaps not surprising that they are satisfied only along a ridge in action space.

Equation (2.5a) states that f_0 will increase where div$\mathbf{F} < 0$ and decrease where div$\mathbf{F} > 0$. The right panel of Figure 3.2 shows the change over some time interval in the DF of an N-body simulation of the disc. We see that the blue region of increase and the red region of decrease broadly coincide with the regions of negative and positive div$\mathbf{F}$.

In a series of carefully controlled N-body simulations, Fouvry et al (2015) checked the predictions of Chapter 2 regarding how the magnitude of $\mathbf{F}$ scales with particle number N and with active mass fraction ξ (which determines the magnitude of $|\mathbf{E}|^2$ and thus $\mathbf{F}$). The flux is predicted to be proportional to m, the mass of a single particle, so it should scale as $1/N$. The experiments indicate that the changes in f_0 induced by the flux follow this scaling within the errors.

From equation (A.7) one can deduce that increasing ξ from 0.5 to 0.6 magnifies $\mathbf{F}$ by a factor 42 while in the N-body experiments this change in ξ increases the change in f_0 in a given period by a factor 29. Given the significant uncertainties in the numerical work, this comparison again amounts to agreement within the errors.

The work of Fouvry et al leaves no doubt that equation (A.7) correctly predicts the action-space flux that Poisson noise drives in a stable but responsive disc, and that until this flux has substantially modified f_0, Poisson noise is the only driver of evolution. The evolution occurs 1000 to 10 000 times faster than naive estimates of two-particle relaxation predict because the noise excites collective motions, which are then amplified by a process called **swing amplification**. Specifically, noise excites leading spiral waves of density that propagate outwards. These waves are gradually unwound by shear in the disc: the angular velocity of star streaming increases inwards, so any structure in the disc is constantly being sheared towards a tightly-wound trailing structure. As a leading-arm spiral is sheared into a trailing-arm structure, self-gravity abruptly amplifies it. The resulting larger-amplitude trailing wave then propagates back inwards. It is Landau damped when it reaches stars that resonate with it. The strength of the swing amplifier determines the magnitude of $\mathbf{E}$, and thus the magnitude of the diffusive flux. Increasing the active mass fraction ξ strengthens the swing amplifier and thus increases $\mathbf{F}$.

Fouvry et al do not compute the evolution of f_0 by integrating equation (2.5a) because computing $\mathbf{F}$ at a single time is already a major task. But the N-body models give insight into what we would find if we did integrate (2.5a), and it's extraordinary. The ridge of enhanced div$\mathbf{F}$ in Figure 3.2 would create a ridge of enhanced f_0, and this ridge would make the disc unstable at a *collisionless* level. That is, Poisson noise drives a disc that is stable towards one that is unstable. Indeed all simulations of initially stable discs that have been integrated for sufficiently long have developed O(1) non-axisymmetries and degenerated into a strong bar. The larger the number N of particles in your disc, the longer you have to wait for the bar to form, but it always does form. Moreover, the final stages in which O(1) spiral structure develops into a bar occur on the same timescale regardless of the value of N – this fact implies that the final-stage dynamics are collisionless. By increasing N you simply increase the delay before the final stage is reached by decreasing the value of $f_1(\mathbf{n}, \mathbf{J}, 0)$ in our equations and thus the initial rate at which f_0 is modified into an unstable DF.

Appendix A: Rewriting $\mathbf{D}_1$

We can bring $\mathbf{F}_1$ to a form that closely parallels eqn (2.17) for $\mathbf{F}_2$. We first note that

$$\sum_{\mathbf{n}} \mathbf{n} E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0) = \sum_{\mathbf{n}}(-\mathbf{n})E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J},-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0),$$

so from (1.33b)

$$\begin{aligned}\mathbf{F}_1(\mathbf{J}) &= -\mathrm{i}m\tfrac{1}{2}\sum_{\mathbf{n}}\mathbf{n}\left[E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0) - E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J},-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)\right]\\ &= \mathrm{i}m\tfrac{1}{2}\sum_{\mathbf{n}}\mathbf{n}\left\{E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J},-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}) - [E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J},-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})]^*\right\}\\ &= \mathrm{i}\frac{m}{2\mathcal{E}}\sum_{\mathbf{n}}\mathbf{n}\sum_{\alpha\alpha'}\hat\Phi^{(\alpha')}(\mathbf{n},\mathbf{J})[\hat\Phi^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\left\{\boldsymbol{\epsilon}^{-1}_{\alpha'\alpha}(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0) - [\boldsymbol{\epsilon}^{-1}_{\alpha\alpha'}(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)]^*\right\},\end{aligned} \tag{A.1}$$

where the second equality uses (2.16). In the curly bracket of the last line we have the difference between $\boldsymbol{\epsilon}^{-1}$ and $\boldsymbol{\epsilon}^{-1\dagger}$. We use

$$\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}^{-1\dagger} = \boldsymbol{\epsilon}^{-1}(\boldsymbol{\epsilon}^\dagger - \boldsymbol{\epsilon})\boldsymbol{\epsilon}^{-1\dagger}. \tag{A.2}$$

From equation (1.32b)

$$\begin{aligned}\Big\{[\boldsymbol{\epsilon}(p)]^\dagger - \boldsymbol{\epsilon}(p)\Big\}_{\beta\beta'} &= -\frac{(2\pi)^3}{\mathcal{E}}\mathrm{i}\int \mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}[\hat\Phi^{(\beta)}(\mathbf{n}',\mathbf{J}')]^*\Phi^{(\beta')}(\mathbf{n}',\mathbf{J}')\\ &\qquad\times\left\{\frac{1}{p^* - \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'} + \frac{1}{p+\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}_0'}\right\}.\end{aligned} \tag{A.3}$$

We need to put $p = \gamma - \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0$ with $\gamma > 0$ and extract the limit $\gamma\to 0$ as per Box 1.2.

$$\begin{aligned}\Big\{[\boldsymbol{\epsilon}(p)]^\dagger - \boldsymbol{\epsilon}(p)\Big\}_{\beta\beta'} &= -\frac{(2\pi)^3}{\mathcal{E}}\mathrm{i}\int \mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}[\hat\Phi^{(\beta)}(\mathbf{n}',\mathbf{J}')]^*\Phi^{(\beta')}(\mathbf{n}',\mathbf{J}')\\ &\qquad\times\left\{\frac{1}{\mathrm{i}(\mathbf{n}\cdot\mathbf{\Omega}_0 - \mathbf{n}'\cdot\mathbf{\Omega}_0')+\gamma} - \frac{1}{\mathrm{i}(\mathbf{n}\cdot\mathbf{\Omega}_0 - \mathbf{n}'\cdot\mathbf{\Omega}_0')-\gamma}\right\}.\end{aligned} \tag{A.4}$$

The principal parts of the two integrals cancel but the contributions from skirting the pole add because in the left integral the pole is at $z = \mathbf{n}\cdot\mathbf{\Omega}_0 - \mathrm{i}\gamma$ and in the right integral it's at $z = \mathbf{n}\cdot\mathbf{\Omega}_0 + \mathrm{i}\gamma$. Glancing back at (A.3) we see that the right integral has exactly the form considered in Box 1.2, so it yields $+\pi K$. The left integral will yield minus this, so

$$\Big\{[\boldsymbol{\epsilon}(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)]^\dagger - \epsilon(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)\Big\}_{\beta\beta'} = -\mathrm{i}\frac{(2\pi)^4}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}\delta(\mathbf{n}'\cdot\mathbf{\Omega}_0' - \mathbf{n}\cdot\mathbf{\Omega}_0)[\hat\Phi^{(\beta)}(\mathbf{n}',\mathbf{J}')]^*\Phi^{(\beta')}(\mathbf{n}',\mathbf{J}'). \tag{A.5}$$

Inserting eqn (A.5) in (A.2) and then in (A.1), we arrive at

$$\begin{aligned}\mathbf{F}_1(\mathbf{J}) &= m\frac{(2\pi)^4}{\mathcal{E}^2}\tfrac{1}{2}\sum_{\mathbf{n}}\mathbf{n}\sum_{\alpha\alpha'\beta\beta'}\hat\Phi^{(\alpha')}(\mathbf{n},\mathbf{J})\epsilon^{-1}_{\alpha'\beta}(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)\int\mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}\\ &\qquad\times\delta(\mathbf{n}'\cdot\mathbf{\Omega}_0' - \mathbf{n}\cdot\mathbf{\Omega}_0)[\hat\Phi^{(\beta)}(\mathbf{n}',\mathbf{J}')]^*\Phi^{(\beta')}(\mathbf{n}',\mathbf{J}')[\epsilon^{-1}_{\alpha\beta'}(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)]^*[\hat\Phi^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\\ &= (2\pi)^4 m\tfrac{1}{2}\sum_{\mathbf{n}}\mathbf{n}\int\mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}\delta(\mathbf{n}'\cdot\mathbf{\Omega}_0' - \mathbf{n}\cdot\mathbf{\Omega}_0)\mathbf{n}|E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)|^2.\end{aligned} \tag{A.6}$$

Comparing equations (2.17) and (A.6) we see that they have extremely similar structures, so when we combine them to form the total flux $\mathbf{F} = \mathbf{F}_2 + \mathbf{F}_1$ we obtain quite a simple bottom line:

$$\begin{aligned}\mathbf{F}(\mathbf{J}) &= (2\pi)^4 m\tfrac{1}{2}\sum_{\mathbf{n}\mathbf{n}'}\mathbf{n}\int\mathrm{d}^3\mathbf{J}'|E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)|^2\delta(\mathbf{n}'\cdot\mathbf{\Omega}_0' - \mathbf{n}\cdot\mathbf{\Omega}_0)\left(\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}f_0(\mathbf{J}) - \mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}f_0(\mathbf{J}')\right)\\ &= (2\pi)^4 m\tfrac{1}{2}\sum_{\mathbf{n}\mathbf{n}'}\mathbf{n}\int\mathrm{d}^3\mathbf{J}'|E_{\mathbf{n}\mathbf{n}}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}_0)|^2\delta(\mathbf{n}'\cdot\mathbf{\Omega}_0' - \mathbf{n}\cdot\mathbf{\Omega}_0)\left(\mathbf{n}'\cdot\frac{\partial}{\partial\mathbf{J}'} - \mathbf{n}\cdot\frac{\partial}{\partial\mathbf{J}}\right)f_0(\mathbf{J})f_0(\mathbf{J}').\end{aligned} \tag{A.7}$$

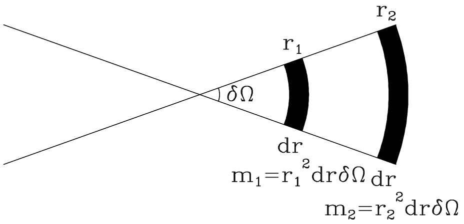

Figure B.1 Each shaded portion of the cone contributes equally to the force on the star at its apex.

Appendix B: Overview of stellar dynamics

We have to deal with particles that may be stars, planets, or dark-matter particles of mass $< 1\,\mathrm{TeV}$. We shall speak of "stars" regardless of whether the particles really are stars.

Since gravitational mass seems to be equal to inertial mass, mass does not appear in an individual particle's equation of motion $\ddot{\mathbf{x}} = -\boldsymbol{\nabla}\Phi$. Particles differing in mass *do* nevertheless have subtly different dynamics because on account of correlations massive particles experience a potential Φ that differs subtly from that experienced by low-mass particles.

In a gas the particles interact by a short-range force, 2 at a time; in a plasma they interact by a long-range force, but shielding suppresses system-wide accumulation of force. In a stellar system the force is long-range and unshielded.

In this regard a stellar system is more complex than a plasma, but it is simpler because relativistic effects can be neglected – in a plasma the cancelling electrostatic forces are so huge that miniscule relativistic corrections to them become crucial – magnetism. In a stellar system one can neglect relativist corrections to the "electrostatic" force Gm_1m_2/r^2.

The contribution to the force on a test star of mass m from stars at distance r in a cone is $\delta\mathbf{F} = [Gm\rho(r)r^2\delta\Omega/r^2]\delta r$, so over distances within which $\rho \sim$ const, equal forces come from equal intervals in distance (Figure B.1). That is the net force is sensitive to the bulk of the system and the contribution of closest neighbours is very small. Hence, the force can be accurately estimated by the **mean-field model** in which the mass of each star is spread through a sphere comparable in size to the inter-star distance; i.e., we compute the potential $\Phi(\mathbf{x})$ and force $-m\nabla\Phi$ from a smooth mass distribution $\rho(\mathbf{x})$.

Our zeroth-order approximation to the motion of a star is the trajectory that follows from its initial $(\mathbf{x}, \mathbf{v})$ and $\Phi(\mathbf{x})$.

B.0.1 Isothermal sphere

Consider the case in which the **distribution function (DF)** is the Maxwellian

$$f(E) = \frac{\mathrm{e}^{-E/\sigma^2}}{(2\pi\sigma^2)^{3/2}} \tag{B.1}$$

with σ a constant and $E(\mathbf{x}, \mathbf{v}) = \frac{1}{2}v^2 + \Phi(|\mathbf{x}|)$ the energy per unit mass. It's natural to assume that the system is spherically symmetric and place the origin at its centre, so the mass density $\rho(r)$ and potential $\Phi(r)$ are functions of $r = |\mathbf{x}|$ only. The density is

$$\begin{aligned}\rho(r) &= (2\pi\sigma^2)^{-3/2}\int \mathrm{d}^3\mathbf{v}\,\mathrm{e}^{-(\frac{1}{2}v^2+\Phi)/\sigma^2} = \frac{4\pi}{(2\pi\sigma^2)^{3/2}}\mathrm{e}^{-\Phi/\sigma^2}\int_0^\infty \mathrm{d}v\,v^2\mathrm{e}^{-v^2/2\sigma^2} \\ &= \mathrm{e}^{-\Phi(r)/\sigma^2}.\end{aligned} \tag{B.2}$$

To find how ρ and Φ vary with r, we need to solve the simultaneous ordinary differential equations

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM(r)}{r^2} \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\rho = 4\pi r^2\mathrm{e}^{-\Phi/\sigma^2} \tag{B.3}$$

One analytic solution is known:

$$\Phi = 2\sigma^2\ln r + \text{constant}, \quad M = \frac{2\sigma^2}{G}r \tag{B.4}$$

In this solution the density $\rho \propto r^{-2}$ diverges at the origin. Solutions with finite central densities can be found by integrating the above equations numerically. At large r all these solutions asymptote to the analytic solution, so in all cases the mass M diverges like r as $r \to \infty$.

It's of the first importance for stellar dynamics that no system of finite mass can have a Maxwellian velocity distribution. Stellar systems with finite mass can be constructed by adopting a DF that falls to zero rather faster than a Maxwellian at large v. For example King models, which provide good fits to many globular clusters, have DFs of the form

$$f(E) = \begin{cases} F(\mathrm{e}^{-E/\sigma^2} - 1) & \text{for } E < 0 \\ 0 & \text{otherwise} \end{cases} \tag{B.5}$$

B.1 The virial theorem

We now prove a very useful result that follows from the scale-free nature of the gravitational interaction. Suppose we differentiate a kind of moment of inertia $I = \sum_\alpha m_\alpha |\mathbf{x}_\alpha|^2$:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2\sum_\alpha m_\alpha \left(\mathbf{x}_\alpha \cdot \ddot{\mathbf{x}}_\alpha + |\dot{\mathbf{x}}_\alpha|^2\right). \tag{B.6}$$

Now

$$\ddot{\mathbf{x}}_\alpha = G\sum_\beta \frac{m_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}(\mathbf{x}_\beta - \mathbf{x}_\alpha) \tag{B.7}$$

so

$$\sum_\alpha \mathbf{x}_\alpha \cdot \ddot{\mathbf{x}}_\alpha = G\sum_{\alpha\beta} \frac{m_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}\mathbf{x}_\alpha \cdot (\mathbf{x}_\beta - \mathbf{x}_\alpha). \tag{B.8}$$

Interchanging α, β on the right side and adding to the original equation, we find

$$2\sum_\alpha \mathbf{x}_\alpha \cdot \ddot{\mathbf{x}}_\alpha = G\sum_{\alpha\beta} \frac{m_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}(\mathbf{x}_\alpha - \mathbf{x}_\beta)\cdot(\mathbf{x}_\beta - \mathbf{x}_\alpha) = -G\sum_{\alpha\beta}\frac{m_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} = 2W, \tag{B.9}$$

where W is the system's potential energy. Inserting this into (B.6)

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2W + 4K, \tag{B.10}$$

where K is the system's kinetic energy. If the system is statistically in a steady state, $I =$ const to within Poisson noise, and we have the **virial theorem**

$$2K + W = 0. \tag{B.11}$$

B.1.1 Negative specific heat

When we eliminate W between the virial theorem and $E = K + W$, we obtain $K = -E$. If we identify K with the temperature T, as is natural, this relation implies for the specific heat $C_V = \partial E/\partial T = -1$. That is a self-gravitating system has negative specific heat: it heats up when you extract energy from it!

Two systems that have negative specific heats cannot come into thermal equilibrium with one another: if system A happens to be slightly hotter than system B, heat will flow out of A into B, raising the temperature of A and depressing the temperature of B. Consequently, more heat will flow and the initially small temperature difference will become large.

Consider a setup in which a stellar system is in thermal contact with an ordinary heat bath of specific heat $C_V > 0$. Let the temperature T_b of the heat bath be initially slightly smaller than the temperature T_s of the stellar system, so heat flows from the from the stellar system to the bath. Then T_s increases by $\delta T_\mathrm{s} = \delta E$ and the temperature of the bath (which gains E) increases by $\delta T_\mathrm{b} = \delta E/C_V$. The transfer of energy will have diminished the temperature difference that drives it only if $\delta T_\mathrm{b} > \delta T_\mathrm{s}$. This condition will hold only if $C_V < 1$. That is, a system with negative specific heat can come into thermal equilibrium with a system with positive specific heat only if the latter is small enough. If you bring a stellar system into contact with a large heat bath, a runaway situation will ensue in which energy flows out of the stellar system faster and faster as the temperature difference between the two bodies grows.

B.2 Fluctuations

We consider a system of mass M and characteristic scale R, in which the characteristic internal speed is $\sigma = \sqrt{GM/R}$. Consider now a subregion of size $r = xR$, which contains mass $M_r \simeq x^3 M$. If there are N stars in the entire system, then $n \simeq x^3 N$ is the typical number of stars in the subregion, and on account of Poisson noise M_r fluctuates by $\delta M_r = M_r/\sqrt{n} = x^3 M/\sqrt{x^3 N}$ during times $\delta t = r/\sigma$. Consider a point that is distance yR from our subregion. At this point a single fluctuation in the subregion's gravitational attraction will change the velocity of a test star by

$$\delta v = \frac{G\delta M_r}{(yR)^2}\delta t = \frac{GMx^{3/2}}{(yR)^2\sqrt{N}}\frac{xR}{\sigma} = \frac{\sigma x^{5/2}}{y^2\sqrt{N}}. \tag{B.12}$$

This formula states that for given y, large volumes $x \simeq 1$ perturb v very much more strongly than small volumes $x \ll 1$. Against this trend we must bear in mind that (a) $y \geq x$, (b) the number of subregions perturbing increases as x^{-3} as x decreases, and (c) the time within which the contribution (B.12) comes about decreases with x, so in a given time each small subregion makes many more contributions to v than does a large subregion.

We assume that the contributions to v from different subregions are statistically independent, so it's appropriate to add the δv in quadrature. There are $\sim 4\pi(y/x)^2$ subregions of scale x that are distance yR from our point, and in a crossing time $t_\mathrm{cross} = R/\sigma$ each such subregion contributes x^{-1} times. So in a crossing time all these subregions change v^2 by

$$(\Delta v)^2 = 4\pi\frac{y^2}{x^3}(\delta v)^2 = 4\pi\frac{\sigma^2 x^2}{y^2 N}. \tag{B.13}$$

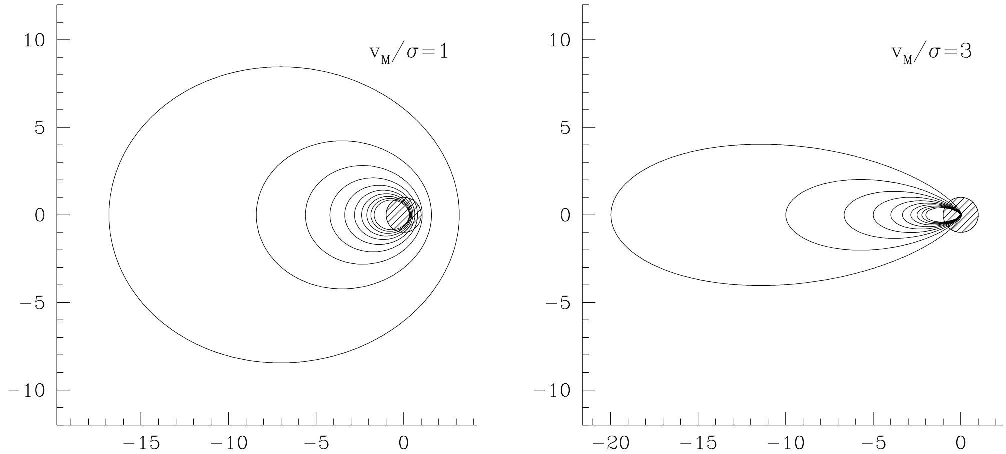

Figure B.2 Contours of overdensity behind a massive body that moves through a sea of field stars with speed equal to the star's velocity dispersion (left) or three times the dispersion (right). The unit of length is GM/σ^2 (From Binney & Tremaine 2008).

Now we have to sum over $y = x, 2x, 3, \ldots, 1$. We convert the sum to an integral using $\mathrm{d}y = x$ and have

$$\sum \frac{1}{y^2} \simeq \frac{1}{x}\int_x^1 \frac{\mathrm{d}y}{y^2} = \frac{1}{x}\left(\frac{1}{x} - 1\right) \simeq \frac{1}{x^2}. \tag{B.14}$$

Hence in a crossing time the subregions of scale x change v^2 by

$$(\Delta v)^2 \simeq 4\pi\sigma^2/N. \tag{B.15}$$

Remarkably, this is independent of x, so regions of each scale xR contribute equally to changing v^2. We obtain the total change by summing these contributions over all relevant values of x. We do this by multiplying equation (B.15) by $-\ln x_{\min}$, where $x_{\min}R$ is the smallest subregion it's sensible to consider. This clearly shouldn't be smaller than a decent multiple of the inter-particle distance $\sim R/N^{1/3}$.

$$(\Delta v)^2_{t_{\rm cross}} \simeq \frac{4\pi\sigma^2 \ln N}{3N} \tag{B.16}$$

The **relaxation time** is the time required for fluctuations to change any velocity by order of itself, thus for $(\Delta v)^2$ to accumulate to σ^2. From (B.16) it follows that

$$t_{\rm relax} \simeq \frac{N}{4\ln N} t_{\rm cross}. \tag{B.17}$$

In an ideal gas the number of molecules in a given volume experiences Poisson fluctuations as was assumed above, and these fluctuations can be considered to arise from thermally excited sound waves. The self-gravity of a stellar system makes the system more compressible on large scales than on small scales, where self gravity is unimportant and an ideal gas provides a valid model. Hence, large-scale fluctuations have a larger amplitude than simple Poisson fluctuations, with the consequence that contrary to our finding above of equal contributions from all scales, fluctuations on the size of the system are dominant. In Chapter 2 we will develop the apparatus required to include the amplifying effect of self gravity, and in Chapter 3 we will see that self-gravity accelerates the relaxation of stellar discs by orders of magnitude. Its effect is much smaller in star clusters.

B.2.1 Dynamical friction

If a stationary mass were to be inserted into a star cluster, it would obviously draw stars to its location, thus increasing the star-density in its vicinity. If the mass were then moved, the region of enhanced density around it would also move, but only after a small delay. Since its region of enhanced background density would be centred on a point that lags the motion of the mass, the gravitational attraction of the region for the mass would pull the mass backwards, just like a frictional force.

In fact every star in a cluster, no matter how massive, experiences such **dynamical friction**: thermal equilibrium involves a balance between stochastic acceleration by fluctuations in the density field, and dynamical friction.

Naturally the magnitude of the overdensity around a particle is proportional to the particle's mass, so the deceleration experienced as a result of dynamical friction is proportional to mass. For drift velocities that are smaller than the mean random velocity in the system, the frictional deceleration is also proportional to the particle's speed because the bigger its speed, the more the overdensity lags the particle's motion. (Figure B.2)

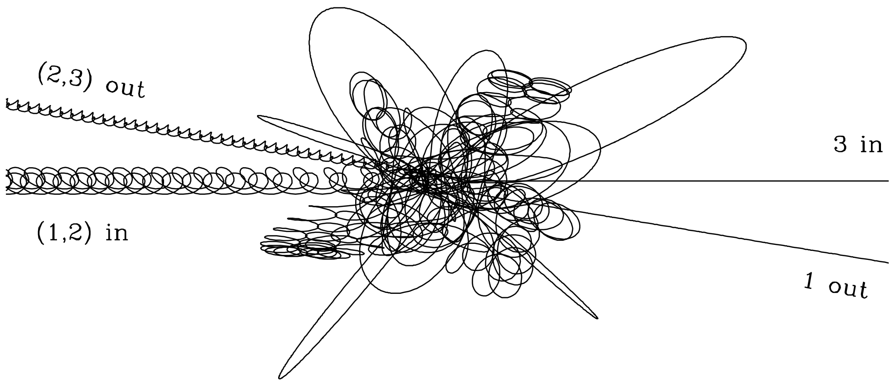

Figure B.3 .7 A close encounter between a binary star and a single star

B.2.2 Equipartition & mass segregation

While dynamical friction is proportional to particle mass, stochastic acceleration affects all bodies equally. Hence the speed at which the two effects come into balance decreases with increasing mass. It follows that in any given volume massive stars will tend to move more slowly. As a consequence, they will tend to accumulate at the centre of a cluster, while the cluster's halo will be anomalously rich in low-mass stars.

B.2.3 Core collapse

We saw above (§B.1.1) that the virial theorem predicts that a self-gravitating system has a specific heat $C_V = -1$. Think of the core of a centrally concentrated stellar system as an independent, self-gravitating system. Fluctuations in the potential allow it to exchange energy with the system's envelope, which will itself have a negative specific heat. Generically the core will contract, heat up and the envelope will expand and cool on absorbing heat from the core. Consequently the temperature difference between the core and the envelope grows steadily. This is the phenomenon of core collapse. Many globular clusters are known in which this apparently apocalyptic process has run to completion.

B.2.4 Evaporation

Encounters in a stellar system will drive $n(\mathbf{v}) = f(\mathbf{x}, \mathbf{v})$ towards a Maxwellian. But stars that reach $v > v_{\rm esc}(\mathbf{x}) = \sqrt{2|\Phi(\mathbf{x})|}$ will escape. By the virial theorem (B.11)

$$\tfrac{1}{2}M\overline{v_{\rm esc}^2} \equiv -\int \mathrm{d}^6\mathbf{w}\, f\Phi = -\int \mathrm{d}^3\mathbf{x}\rho\Phi = -2\mathrm{PE} = 2\int \mathrm{d}^6\mathbf{w}\, fv^2 \equiv 2M\overline{\sigma}^2 \tag{B.18}$$

so

$$\overline{v_{\rm esc}} = 2\overline{\sigma} \tag{B.19}$$

In a Maxwellian a fraction $\sim 1/135$ of stars have speeds greater than 2σ. Hence the system will lose 1/135 of its mass each relaxation time by **evaporation** of stars.

B.3 Binary stars

To this point our strategy has been to ignore correlations between stars beyond those implied by Poisson fluctuations – we have characterised the system by its one-particle distribution function. However, a significant fraction of stars are members of a binary system and binary stars can have a big impact on the long-term evolution of star clusters.

When a field star has a close encounter with a binary star, energy is exchanged between the binary's internal energy $E_{\rm b} < 0$ and the translational KE of the field and binary stars. If $|E_{\rm b}|$ is decreased too much, the binary is disrupted (ionised). If $|E_{\rm b}|$ is increased, the cluster is heated.

B.3.1 Soft binaries

If $E_{\rm b}| < \frac{1}{2}m\sigma^2$ the binary is **soft**. The internal speed of the binary is slower than the random motion in the cluster. Given that an intruder will pass through the binary in only a fraction of a binary period, it will affect the binary only if it passes much closer to one member than the binary separation. So it essentially scatters just one member. On average, this scattering will soften, and may disrupt, the binary.

B.3.2 Hard binaries

If $|E_{\rm b}| > \frac{1}{2}m\sigma^2$ the binary is **hard**. When a field star comes close to a hard binary, an unstable triple star is formed (Figure B.3). Eventually one of the three stars is ejected with a velocity comparable to the original binary velocity, so faster than σ. By conservation of energy the final binary is harder than the original binary: **Heggie's law**: hard binaries become harder, soft binaries softer.

The bottom line: hard binaries are an energy source for a cluster just as nuclear fusion is an energy source for stars. Soft binaries come and go without significant impact. The field star may exchange places with one of the binary stars.

Formation of hard binaries A binary can form in a cluster of point masses only through the interaction of 3 bodies: one is there to carry away the energy released as the other two form a binary. If a hard binary is to form, the relative velocity of the future binary members has to change by of order itself, so the impact parameter has to be not much larger than

$$b_{90} \equiv \frac{Gm}{v_\infty^2}, \tag{B.20}$$

where v_∞ is the relative velocity with which the two stars approach one another. If the density of stars is n, the characteristic time interval t_{90} between a given star experiencing a large deflection is given by $nb_{90}^2 v_\infty t_{90} \simeq 1$, and the probability that when this encounter occurs there is a third particle within b_{90} is $P_3 \simeq nb_{90}^3$. Hence the time required for a given star to have a non-negligible probability of entering a tight binary is

$$t_2 \simeq \frac{t_{90}}{P_3} \simeq \frac{1}{n^2 b_{90}^5 v_\infty} \simeq \frac{1}{n^2 v_\infty}\left(\frac{v_\infty^2}{Gm}\right)^5 \tag{B.21}$$

But by the virial theorem (B.11), the cluster radius $R \sim GNm/\sigma^2$ so taking $v_\infty \simeq \sigma$ and using with (B.17)

$$\frac{t_2}{t_{\rm relax}} \simeq \frac{1}{n^2\sigma}\frac{N^5}{R^5}\frac{4\ln N\sigma}{NR} \simeq \frac{R^6}{N^2\sigma}\frac{1}{R^5}\frac{4\ln N^5\sigma}{NR} = 4N^2\ln N \tag{B.22}$$

The time required for one of the N stars in the core to get into a binary is smaller than t_2 by a factor N and the first tight binary will form after

$$t_{\rm fst} \sim t_2/N \simeq 4N\ln N t_{\rm relax}. \tag{B.23}$$

Problems

1 Write down the generating function $S(\boldsymbol{\theta}, \mathbf{J}')$ of the canonical transformation $(\boldsymbol{\theta}, \mathbf{J}) \leftrightarrow (\boldsymbol{\theta}', \mathbf{J}')$ that makes ordinary phase-space coordinates periodic in the θ'_i with period unity rather than 2π.

2 Let $S(x, J)$ be the generating function of the canonical transformation between (x, p) and the angle-action coordinates of the harmonic oscillator $H(x, p) = \frac{1}{2}(p^2 + \omega^2 x^2)$. Explain what the Hamilton-Jacobi equation is, and show that for this system it yields

$$S = \frac{E}{\omega}\left(\psi + \tfrac{1}{2}\sin 2\psi\right), \tag{P.24}$$

where $\sin\psi \equiv \omega x/\sqrt{2E}$. Define the action and show that for this system it is $J = E/\omega$. Hence show that

$$S = J\left(\psi + \tfrac{1}{2}\sin 2\psi\right), \tag{P.25}$$

Hence show that $\theta = \psi$

3 Particles move in the (r, ϕ) plane in the potential $\Phi(r)$. Write down the Hamilton-Jacobi equation for the generating function $S(r, \phi, J_r, J_\phi)$. By writing $S = S_r(r, J_r, J_\phi) + S_\phi(\phi, J_r, J_\phi)$ show that $J_\phi = p_\phi$ and obtain an integral for J_r. Show that

$$\theta_r(r, \mathbf{J}) = \Omega_r \int \frac{\mathrm{d}r}{p_r} \tag{P.26}$$

where Ω_r is the radial frequency. Give a physical interpretation of this result.

4 N particles form a system with Hamiltonian

$$H = \tfrac{1}{2}\sum_i \left(\mathbf{p}_i^2 + \sum_j u(\mathbf{q}_i, \mathbf{q}_j)\right), \tag{P.27}$$

where u is a symmetric function of its arguments. Show from first principles that the N-particle DF satisfies

$$\frac{\partial f^{(N)}}{\partial t} + [f^{(N)}, H] = 0, \tag{P.28}$$

where the Poisson bracket $[f, g]$ is defined by

$$[f, g] = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial \mathbf{q}_i}\cdot\frac{\partial g}{\partial \mathbf{p}_i} - \frac{\partial f}{\partial \mathbf{p}_i}\cdot\frac{\partial g}{\partial \mathbf{q}_i}\right). \tag{P.29}$$

Explain why equation (P.28) can be written $\mathrm{d}f^{(N)}/\mathrm{d}t = 0$.

5 A good orthogonal potential-density basis $(\Phi^{(\mathbf{n})}, \rho^{(\mathbf{n})})$ starts with the **Hernquist sphere**

$$\rho^{(0)}(r) = \frac{\rho_0}{r/a(1 + r/a)^3} \quad \leftrightarrow \quad \Phi^{(0)}(r) = -\frac{2\pi G\rho_0 a^2}{1 + r/a}, \tag{P.30}$$

where ρ_0 and a are constants. Show that in this case the constant $\mathcal{E} = GM^2/a$, where $M = 2\pi\rho_0 a^3/\sqrt{3}$ is the system's mass. Explain why it's reasonable to adopt as other members of the family

$$\Phi^{(0,l,m)} = \frac{(r/a)^l C}{(1 + r/a)^{2l+1}} \mathrm{Y}_l^m(\theta, \phi), \tag{P.31}$$

where C is a constant to be determined. Show that the corresponding density distribution is

$$\rho^{(0,l,m)} = -\frac{(2l + 1)(l + 1)C}{2\pi G a^2}\frac{(r/a)^{l-1}}{(1 + r/a)^{2l+3}}\mathrm{Y}_l^m(\theta, \phi).$$

Explain how the constant C is determined. (Much more detail in Hernquist & Ostriker, ApJ, 386, 375 (1992))

6 Show that the action-space flux $\mathbf{F}$ defined by equation (2.6) is necessarily real.

7 Derive from Liouville's equation for the full N-particle DF $f^{(N)}(\mathbf{x}_1,\ldots,\mathbf{v}_N)$ the Boltzmann eq that connects the 1-particle and 2-particle DFs $f^{(1)}$ and $f^{(2)}$.

8 Show from the definition (1.32b) $\boldsymbol{\epsilon}$ that

$$[\boldsymbol{\epsilon}^{-1}(p)]^* = \boldsymbol{\epsilon}^{-1}(p^*).$$

9 Using the result of Appendix A, show that the diffusive flux $\mathbf{F}$ vanishes when the DF is $f_0(\mathbf{J}) \propto \mathrm{e}^{-\beta H_0(\mathbf{J})}$, where β is a constant. What physical principle does this result vindicate/illustrate?

10 Let $f_0(\mathbf{J})$ be the distribution function (DF) of an equilibrium stellar system that has gravitational potential $\Phi_0(\mathbf{x})$ and angle-action coordinates $(\boldsymbol{\theta},\mathbf{J})$. Show that if we write the DF of the perturbed model $f(\mathbf{x},\mathbf{v},t) = f_0 + f_1(\mathbf{x},\mathbf{v},t)$, then to first order in the perturbations f_1 satisfies

$$\frac{\partial f_1}{\partial t} + \boldsymbol{\Omega}_0 \cdot \frac{\partial f_1}{\partial \boldsymbol{\theta}} - \frac{\partial f_0}{\partial \mathbf{J}} \cdot \frac{\partial \Phi_1}{\partial \boldsymbol{\theta}} = 0, \tag{P.32}$$

where $\boldsymbol{\Omega}_0 = \partial H_0/\partial \mathbf{J}$ and the perturbed potential is $\Phi(\mathbf{x},t) = \Phi_0(\mathbf{x}) + \Phi_1(\mathbf{x},t)$. Hence or otherwise show that

$$\widetilde{f}_1(\mathbf{n},\mathbf{J},p) = \frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p) + \hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0}, \tag{P.33}$$

where the meanings of a tilde and a hat should be explained.

What physical principle is used to obtain from the last equation the expression

$$\widetilde{\Phi}_1(\mathbf{n}',\mathbf{J}',p) = -(2\pi)^3 \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{n}} E_{\mathbf{n}'\mathbf{n}}(\mathbf{J}',\mathbf{J},p)\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0}, \tag{P.34}$$

where $\mathbf{E}$ is the inverse of the "dielectric tensor"? Explain (without calculation) how from this equation we can obtain

$$\widetilde{f}_1(\mathbf{n},\mathbf{J},p) = -(2\pi)^3\mathrm{i}\frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0}\int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p)\frac{\hat{f}_1(\mathbf{n}',\mathbf{J}',0)}{p + \mathrm{i}\mathbf{n}'\cdot\boldsymbol{\Omega}_0'} + \frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}_0}. \tag{P.35}$$

Fluctuations in Φ drive a diffusive flux $\mathbf{F}$ of the mass-bearing stars through phase space. $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{J}) = \mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}}\mathrm{e}^{pt}\widetilde{f}_1(\mathbf{n},\mathbf{J},p) \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}}\mathrm{e}^{p't}\widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p')\right\rangle, \tag{P.36}$$

where $\langle\cdot\rangle$ indicates an ensemble average. A population of massless tracer particles orbits within the stellar system. Let $g_0(\mathbf{J})$ and $g_1(\mathbf{x},\mathbf{v},t)$ be the unperturbed and perturbed DFs of this population. Show that the phase-space flux $\mathbf{G}$ of the tracer population is given by an expression of the form (an expression for $\mathbf{D}_2$ is *not* required)

$$\mathbf{G} = -\mathbf{D}_2(\mathbf{J}) \cdot \frac{\partial g_0}{\partial \mathbf{J}}. \tag{P.37}$$

Explain the physical significance of the form taken by $\mathbf{G}$.